Math 259: Spring 2020
Exam 1

NAME: SOLUTIONS

Time: **50 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	12	
2	6	
3	12	
4	13	
5	7	
GC	5	
TOTAL	50 (or 55)	

Problem 1 : (12 points)

a) (6 points) Give the definition of a closed set.

Let X be a topological space. $A \subset X$ is closed if $U = X - A$ is open.

b) (6 points) Show that a finite union of closed sets is closed.

Let $A_1, A_2 \dots A_n$ be closed, so $U_i = X - A_i$ $i=1, \dots n$ are open. Then

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i) = \bigcap_{i=1}^{n} U_i \quad \text{is open}$$

since a finite intersection of open sets is open.

So $\bigcup_{i=1}^{n} A_i$ is closed.

Problem 2 : (6 points) Let X be a topological space and $A \subset X$. Give the definition of a limit point of A. (You may also give any criteria that is equivalent to the definition.)

$x \in X$ is a limit point of A if every neighborhood of x intersects A in a point different from x.

Problem 3 : (12 points)

a) (6 points) Give the definition of a topology on a set X.

Let X be a set. A collection $\mathcal{T}$ of subsets of X is a topology if

① $\phi, X \in \mathcal{T}$

② For any indexing set J, if $U_\alpha \in \mathcal{T}$ for $\alpha \in J$ then $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$

③ If $U_1, \dots, U_n \in \mathcal{T}$ then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

b) (6 points) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Show that $\mathcal{T}'' = \mathcal{T} \cap \mathcal{T}'$ is also a topology on X.

① Since $\phi \in \mathcal{T}$ and $\mathcal{T}'$, $\phi \in \mathcal{T} \cap \mathcal{T}'$

$X \in \mathcal{T}$ and $\mathcal{T}'$, $X \in \mathcal{T} \cap \mathcal{T}$

② Let J be an indexing set and $U_\alpha \in \mathcal{T}' \cap \mathcal{T}$ for $\alpha \in J$.

Then $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$ since each $U_\alpha \in \mathcal{T}$

$\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}'$ since each $U_\alpha \in \mathcal{T}'$

so $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T} \cap \mathcal{T}'$

③ Let $U_1, \dots, U_n \in \mathcal{T} \cap \mathcal{T}'$.

Then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ and $\bigcap_{i=1}^{n} U_i \in \mathcal{T}'$ so

$\bigcap_{i=1}^{n} U_i \in \mathcal{T} \cap \mathcal{T}'$.

Problem 4 : (13 points)

a) (6 points) Let X and Y be topological spaces. Give the definition of a continuous function $f\colon X \to Y$.

Whenever $V \subset Y$ is open, $f^{-1}(V) \subset X$ is open.

b) (7 points) Let X and Y be topological spaces, and consider the space $X \times Y$ with the product topology. Let $\pi_1\colon X \times Y \to X$ be the projection map. Show that π_1 is a continuous map.

Let $U \subset X$ be open. Then $\pi_1^{-1}(U) = U \times Y$ which is also open.

Problem 5 : (7 points) Let X be a Hausdorff space. Consider the space $X \times X$ with the product topology, and its subset

$$\Delta = \{x \times x \in X \times X \mid x \in X\}$$

called the *diagonal of* X. Show that Δ is closed in $X \times X$.

We show that $\bar{\Delta} = \Delta$.

Let $x \times y \in \bar{\Delta}$.

If $x \neq y$, since X is Hausdorff, there are

$U \subset X$ open with $x \in U$
$V \subset X$ open with $y \in V$
and $U \cap V = \phi$

Now consider $U \times V \subset X \times X$. This is open in the product topology, and $x \times y \in U \times V$.

Since $x \times y \in \bar{\Delta}$, $(U \times V) \cap \Delta \neq \phi$. Say $z \times z \in \Delta$ is such that $z \times z \in U \times V$. But then $z \in U \cap V$, a contradiction.

So if $x \times y \in \bar{\Delta}$, $x = y$ and $\bar{\Delta} = \Delta$ so Δ is closed.

Extra problem for graduate credit:

Problem 6 : (5 points) Let X be a simply ordered set with the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold?

Suppose that $x < a$.

If there is $c \in X$ with $c < x$, let $U = (c, a)$, otherwise let $U = [x, a)$. Then U is open, $x \in U$ but $U \cap (a,b) = \phi$. So $x \notin \overline{(a,b)}$.

Suppose that $x > b$.

If there is $c \in X$ with $c > x$, let $U = (b, c)$, otherwise let $U = (b, x]$. Then U is open, $x \in U$ but $U \cap (a,b) = \phi$. so $x \notin \overline{(a,b)}$

Therefore $x \in \overline{(a,b)} \Rightarrow a \leq x \leq b$ so

$$\overline{(a,b)} \subset [a,b]$$

Equality holds if and only if there are no $c, d \in X$ with $a < c \leq d < b$ such that $(a,b) = [c,d]$.

(or in other words, there are no $c, d \in X$ with

$(a,c) = \phi$

$(d,b) = \phi$)