X topological space, $A \subset X$

Suppose that for all $x \in A$ $\exists$ U_x open in X
and $x \in U_x \subset A$

Show A is open.

Claim: $A = \bigcup_{x \in A} U_x$

Note: Once we show this, we are done, because then
A will be a union of open sets and hence open.

Claim: $A = \bigcup_{x \in A} U_x$

① $A \subset \bigcup_{x \in A} U_x$ because if $x \in A$ then $x \in U_x$

so $x \in \bigcup_{x \in A} U_x$

② $\bigcup_{x \in A} U_x \subset A$ because if $y \in \bigcup_{x \in A} U_x$

then $\exists\, x \in A$ such that $y \in U_x$

But $U_x \subset A$ so $y \in A$.