Math 259 - Spring 2019
Homework 3

This homework is due on Wednesday, February 27.

1. For each of the following continued fractions, compute all convergents and the value of the continued fraction. Please do not use a calculator.

   (a) $[1; 1, 2, 3]$
   (b) $[0; 1, 1, 2, 3]$
   (c) $[1; 1, 2, 2, 1]$
   (d) $[2; 4, 1, 2]$
   (e) $[0; 2, 4, 1, 2]$

2. Compute the continued fraction expansions for the following numbers. You may use a calculator or computer for this problem.

   (a) e
   (b) $\sqrt{2}$
   (c) $\frac{1+\sqrt{5}}{2}$

3. Among the convergents of $\sqrt{15}$, find a rational number that approximates $\sqrt{15}$ to four decimal places. You may use a calculator for this problem.

4. (TW Section 19.4, Problem 3)

   (a) Suppose that $\frac{M}{r}$ and $\frac{M_1}{r_1}$ are two distinct rational numbers, with $0 < r < N$ and $0 < r_1 < N$. Show that
   $$\left| \frac{M_1}{r_1} - \frac{M}{r} \right| > \frac{1}{N^2}.$$

   (b) Suppose, as in Shor's algorithm, that we have
   $$\left| \frac{j}{2^q} - \frac{M}{r} \right| < \frac{1}{2N^2} \quad \text{and} \quad \left| \frac{j}{2^q} - \frac{M_1}{r_1} \right| < \frac{1}{2N^2}.$$
   Show that $\frac{M}{r} = \frac{M_1}{r_1}$.

5. Let $\mathcal{F}$ denote the quantum Fourier transform, which acts on the n-qbit superposition $|k\rangle$ by
   $$\mathcal{F}|k\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{\frac{2\pi i}{2^n} jk} |j\rangle,$$
   and acts on the more general n-qbit superposition $\sum_{k=0}^{2^n-1} a_k|k\rangle$ by extending linearly. It is a fact then that
   $$\mathcal{F}\left(\sum_{k=0}^{2^n-1} a_k|k\rangle \right) = \sum_{j=0}^{2^n-1} b_j|j\rangle,$$
   where b_j is the jth Discrete Fourier Transform coefficient of the sequence $\{a_k\}$.

(a) Prove this fact when $n = 2$. Hint: It's not so bad to just expand everything out, there's no need to be clever.

(b) The quantum Fourier transform is a quantum gate. Still when $n = 2$, write it as a matrix.

6. (adapted from TW Section 19.4, Problem 1) Suppose that you are using Shor's algorithm to compute the multiplicative order of 2 modulo 15. Throughout this problem we will use the notation set up in class (which will appear in the notes shortly).

   (a) What value of q would you take?

   (b) Suppose that your measurement in Shor's algorithm is $j = 192$. What value would you obtain for r, the multiplicative order of 2 modulo 15? Does it agree with the real multiplicative order of 2 modulo 15?

   (c) Use your value of r to factor 15.

7. Nevermind, there won't be a problem 7.

Extra problems for graduate credit:

1. (a) Let $x \in \mathbb{R}$. Prove that

$$\sqrt{x} = 1 + \frac{x-1}{1+\sqrt{x}}.$$

   (b) Use this to give a formula for a (generalized, i.e. not simple) continued fraction expansion expression for $\sqrt{x}$.

   (c) Use your formula to check your continued fraction expansion for $\sqrt{2}$ above, and to compute a generalized continued fraction expansion for $\sqrt{3}$.

2. Recall the quantum Fourier transform from problem 4. above.

   (a) Prove that for any n,

$$\mathcal{F}\left(\sum_{k=0}^{2^n-1} a_k|k\rangle\right) = \sum_{j=0}^{2^n-1} b_j|j\rangle,$$

   where b_j is the jth Discrete Fourier Transform coefficient of the sequence $\{a_k\}$.

   (b) Write the quantum Fourier transform as a matrix.