Math 255 - Spring 2018
Homework 1 Solutions

1. We do this proof by induction.

   The base case is $n = 0$: In that case we have
$$\frac{(2 \cdot 0)!}{2^0 0!} = \frac{0!}{1 \cdot 1} = \frac{1}{1} = 1 \in \mathbb{Z}.$$
   Therefore the assertion is true for $n = 0$.

   We now do the inductive step. In this step, we assume that for some integer $k \geq 0$, the quantity
$$\frac{(2k)!}{2^k k!}$$
   is an integer.

   We now investigate the quotient
$$\frac{(2(k+1))!}{2^{k+1}(k+1)!}.$$
   We have the following facts:
$$\begin{aligned}
(2k+2)! &= (2k+2)(2k+1)(2k)! \\
2^{k+1} &= 2 \cdot 2^k \\
(k+1)! &= (k+1)k!
\end{aligned}$$
   This allows us to establish the following equalities:
$$\begin{aligned}
\frac{(2(k+1))!}{2^{k+1}(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{2 \cdot 2^k (k+1)k!} \\
&= \frac{(2k+2)(2k+1)}{2(k+1)} \frac{(2k)!}{2^k k!} \\
&= (2k+1)\frac{(2k)!}{2^k k!}
\end{aligned}$$
   We have that both $\frac{(2k)!}{2^k k!} \in \mathbb{Z}$ (by the induction hypothesis) and $2k+1 \in \mathbb{Z}$ (since the product and sum of integers is an integer), it follows that
$$\frac{(2(k+1))!}{2^{k+1}(k+1)!} \in \mathbb{Z}$$
   since the product of two integers is an integer.

   Therefore the assertion is proved for all values of $n \geq 0$ by induction.

2. We begin with the difficult direction: Assume that $\binom{n}{r} = \binom{n}{r+1}$. By definition of the binomial coefficient, we have therefore

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(r+1)!(n-(r+1))!}. \tag{1}$$

We make two observations to take the next step: First, we have

$$(r+1)! = r!(r+1)$$

and secondly we have

$$(n-(r+1))! = (n-r-1)!$$

and

$$(n-r)! = (n-r-1)!(n-r).$$

Plugging these into equation (1), we get instead

$$\frac{n!}{r!(n-r-1)!(n-r)} = \frac{n!}{(r+1)r!(n-r-1)!}.$$

Multiplying both sides by

$$\frac{r!(n-r-1)!}{n!},$$

we get

$$\frac{1}{n-r} = \frac{1}{r+1}.$$

Since equal fractions that have equal numerators must also have equal denominators, we get that

$$n-r = r+1,$$

from which it finally follows that

$$n = 2r+1.$$

We obtain at once that n is odd, and solving for r, that $r = \frac{1}{2}(n-1)$.

We now tackle the easy direction. We assume that n is odd, and $r = \frac{1}{2}(n-1)$. We note that since n is odd, r is an integer. Solving for n, we get $n = 2r+1$.

Since $n = 2r+1$, we get that

$$n-r = 2r+1-r = r+1,$$

and

$$n-(r+1) = 2r+1-r-1 = r.$$

Therefore we have

$$\begin{aligned}
\binom{n}{r} &= \frac{n!}{r!(n-r)!} \\
&= \frac{n!}{(n-(r+1))!(r+1)!} \\
&= \binom{n}{r+1}.
\end{aligned}$$

3. Beginning with $D = 5$ and using the first column of the addition (the units), we conclude that $T = 0$. We also note that there is a carry into the second column (the columns of the tens).

   We now look at the fifth column of the addition. There are four different scenarios to consider, since there may or may not be a carry from the previous column and there may or may not be a carry to the next column. Here are the cases to consider:

   - No carries at all: $E + O = O$. This would mean that $E = 0$, which is impossible since $T = 0$.
   - Carry from the previous column, but no carry into the next column: $1+E+O = O$. This implies that $E = -1$, which is impossible since that is not a digit.
   - No carry from the previous column, but a carry into the next column: $E + O = O + 10$. This implies that $E = 10$, which is impossible since that is not a digit.
   - Therefore it must be the case that there is both a carry from the previous column and a carry into the next column: $1 + E + O = O + 10$. This implies that $E = 9$.

   We now turn our attention to the last column. We note first that there is no carry into the next column, because there is no next column. We also know that there is a carry from the previous column. Therefore the last column gives us the equation $1 + G + 5 = R$. Since G cannot be 0, R must be 7, 8 or 9, but R cannot be 9 since $E = 9$. Therefore R is 7 or 8. We use the second column to determine the parity of R. Since we know there is a carry from the previous column, there are two cases to consider for the second column:

   - No carry into the next column: $1 + L + L = R$. In that case R is odd.
   - A carry into the next column: $1 + L + L = R + 10$. Again R is odd.

   Therefore in fact $R = 7$. Plugging back into the equation $1 + G + 5 = R$ given by the last column, we get that $G = 1$.

   We now consider the third column of the addition, remembering that $E = 9$. We don't have any carry information for this column, so we consider all four cases again:

   - No carries at all: $A + A = 9$. This has no solution in the integers.
   - Carry from the previous column, but no carry into the next column: $1+A+A = 9$. This implies that $A = 4$.
   - No carry from the previous column, but carry into the next column: $A+A = 9+10$. This has no solution in the integers.
   - Carry from the previous column and carry into the next column: $1+A+A = 9+10$. This implies that $A = 9$, which is impossible since $E = 9$ already.

Therefore $A = 4$, and there is no carry into the fourth column.

We also just now figured out that there must have been a carry from the second column to the third column. Therefore going back to the second column, we now know that the correct equation to consider is $1 + L + L = R + 10$, with $R = 7$, so $L = 8$.

We are now left with the digits $2, 3, 6$ and the letters O, N, B. We cannot get any information about the letter O from this puzzle: It only appears in column five, and the equation $1+E+O = O+10$ does not give us any information about O. Therefore we will figure out this letter last, it will be the last digit that is left.

To figure out N and B, we notice that they both appear in column four. We know that there is no carry from the third column and there is a carry to the fifth column. Therefore we have the equation $7 + N = B + 10$, or $N = B + 3$. Among the digits that are left, the only ones with this relationship force $B = 3$ and $N = 6$.

Therefore $O = 2$, and the addition problem was $197,485 + 526,485 = 723,970$.