Math 395 - Fall 2021
Qual problem set 9

This homework is "due" on Monday November 8 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let $K = \mathbb{Q}(\sqrt{3+\sqrt{5}})$.
   (a) Show that $K/\mathbb{Q}$ is a Galois extension.
   (b) Determine the Galois group of $K/\mathbb{Q}$.
   (c) Find all subfields of K.

2. Let $\alpha = \sqrt{1-\sqrt[3]{5}} \in \mathbb{C}$ (where $\sqrt[3]{5}$ denotes the real cube root), let K be the splitting field of the minimal polynomial of α over $\mathbb{Q}$, and let $G = \mathrm{Gal}(K/\mathbb{Q})$.
   (a) Find the degree of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.
   (b) Show that K contains the splitting field of $x^3 - 5$ over $\mathbb{Q}$ and deduce that G has a normal subgroup H such that $G/H \cong S_3$.
   (c) Show that the order of the subgroup H in (b) divides 8.

3. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let $\alpha = \sqrt{2} - \sqrt{3}$.
   (a) Show that $[L(\sqrt{\alpha}) : L] = 2$ and $[L(\sqrt{\alpha}) : \mathbb{Q}] = 8$.
   (b) Find the minimal polynomial of $\sqrt{\alpha}$ over $\mathbb{Q}$.
   (c) Show that $L(\sqrt{\alpha})$ is not Galois over $\mathbb{Q}$.

4. Let α be the real, positive fourth root of 5, and let $i = \sqrt{-1} \in \mathbb{C}$. Let $K = \mathbb{Q}(\alpha, i)$.
   (a) Prove that $K/\mathbb{Q}$ is a Galois extension with Galois group dihedral of order 8.
   (b) Find the largest abelian extension of $\mathbb{Q}$ in K (i.e., the unique largest subfield of K that is Galois over $\mathbb{Q}$ with abelian Galois group) – justify your answer.
   (c) Show that $\alpha + i$ is a primitive element for $K/\mathbb{Q}$.