Math 395 - Fall 2021
Qual problem set 7

This homework is "due" on Monday October 18 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let p be a prime and let P be a nonabelian group of order p^3.
   (a) Prove that the center of P has order p, i.e., that $\#Z(P) = p$.
   (b) Prove that the center of P equals the commutator subgroup of P, i.e., $Z(P) = P'$.
2. In this problem, G is a finite group.
   (a) Show that if $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G, then G is abelian.
   (b) Let p be a prime and P be a p-group. Show that $Z(P)$ is nontrivial.
   (c) Show that if P has order p^2 then P is abelian.
   (d) Show that every p-group is solvable.
3. Let P be a finite p-group for p a prime, and suppose that P acts on a finite set S.
   (a) Denote by S^P the set of elements of S that are fixed by every element of P:
   $$S^P = \{s \in S : g \cdot s = s \text{ for all } g \in P\}.$$
   Prove that
   $$\#S \equiv \#S^P \pmod{p},$$
   where # denotes the cardinality of the set following it.
   (b) Suppose now that P acts transitively on S. Prove that $\#S$ is a power of p.
4. Let G be a finite group, let N be a normal subgroup of G, and let H be any subgroup of G.
   (a) Prove that if the index of N in G is relatively prime to the order of H, then $H \subseteq N$.
   (b) Prove that if H is any Sylow p-subgroup of G for some prime p, then $H \cap N$ is a Sylow p-subgroup of N.