Math 395 - Fall 2021
Qual problem set 6

This homework is "due" on Monday October 11 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let G be a group of odd order and let σ be an automorphism of G of order 2.
   (a) Prove that for every prime p dividing the order of G there is some Sylow p-subgroup P of G such that $\sigma(P) = P$ (i.e., σ stabilizes the subgroup P – note that σ need not fix P elementwise).
   (b) Suppose that G is a cyclic group. Prove that $G = A \times B$ where
   $$A = C_G(\sigma) = \{g \in G : \sigma(g) = g\} \quad \text{and} \quad B = \{x \in G : \sigma(x) = x^{-1}\}.$$
   (Remark: This decomposition is true more generally when G is abelian.)
2. Let G be a group of order 63.
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for all primes p dividing 63.
   (b) Show that if the Sylow 3-subgroup of G is normal, then G is abelian.
   (c) Let H be a group of order 9. Show that there is only one nontrivial action of the group H on the group C_7 (up to automorphisms of H).
   (d) Show that there are exactly four isomorphism classes of groups of order 63.
3. Fix p a prime and let G be the group of matrices of the form
   $$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$
   where $a, b, c \in \mathbb{F}_p$, and where the operation is multiplication.
   (a) Prove that the subgroup H of matrices where $a = c = 0$ is normal.
   (b) Express the group G/H as a direct product of cyclic groups. (Your answer can be a single cyclic group if G/H is cyclic.) You must justify your answer.
   (c) Prove that for each prime p, there is a group of order p^3 that is not abelian. (You might remember that all groups of order p and p^2 are abelian, when p is prime. It ends there!)

4. Let G be the group $G = \langle a, b \,|\, a^3 = b^4 = bab^{-1}a = 1 \rangle$.

(a) Show that G is a nonabelian group of order 12.

(b) Show that G is not isomorphic to A_4.

(c) Show that G is not isomorphic to D_6, the dihedral group with 12 elements.