Math 395 - Fall 2021
Qual problem set 4

This homework is "due" on Monday September 27 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let G be a group and let H be a subgroup of finite index $n > 1$ in G. Let G act by left multiplication on the set of all left cosets of H in G.
   (a) Prove that this action is transitive.
   (b) Find the stabilizer in G of the identity coset $1H$.
   (c) Prove that if G is an infinite group, then it is not a simple group.

2. Let G be a finite group of order n and let $\pi\colon G \to S_n$ be the (left) regular representation of G into the symmetric group on n elements.
   (a) Prove that if n is even, then G contains an element of order 2. (Do not use Cauchy's Theorem; please prove this directly.)
   (b) Suppose that n is even and x is an element of G of order 2. Prove that $\pi(x)$ is the product of $n/2$ transpositions.
   (c) Prove that if $n = 2m$ where m is odd, then G has a normal subgroup of index 2.

3. (a) Show that S_3 acts transitively on 6 elements by giving an explicit example.
   (b) Any transitive action of S_3 on a set with 6 elements gives an injective group homomorphism $S_3 \hookrightarrow S_6$. For the action you have given in part 4(a), give this homomorphism explicitly.
   (c) Consider the "usual" injective group homomorphism $S_3 \hookrightarrow S_6$ given by sending $(12) \mapsto (12)$ and $(123) \mapsto (123)$. If H_1 is the image of S_3 in S_6 under the homomorphism of part 4(b), and H_2 is the image of S_3 in S_6 under the "usual" injective homomorphism, are H_1 and H_2 conjugate in S_6? Briefly justify your answer.

4. Let G be a *solvable* group of order $168 = 2^3 \cdot 3 \cdot 7$. The aim of this exercise is to show that G has a normal Sylow p-subgroup for some prime p. Let M be a minimal normal subgroup of G.
   (a) Show that if M is not a Sylow p-subgroup for any prime p, then $\#M = 2$ or 4. (You may quote without proof any result you need about minimal normal subgroups of solvable groups.)

(b) Assume that $\#M = 2$ or 4 and let $\overline{G} = G/M$. Prove that $\overline{G}$ has a normal Sylow 7-subgroup.

(c) Under the same assumptions and notations as (b), let H be the complete preimage in G of the normal Sylow 7-subgroup of $\overline{G}$. Prove that H has a normal Sylow 7-subgroup P, and deduce that P is normal in G.