Math 395 - Fall 2021
Qual problem set 2

This homework is "due" on Monday September 13 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let G be a finite group acting transitively (on the left) on a nonempty set Ω. For $\omega \in \Omega$, let G_ω be the usual stabilizer of the point ω:
$$G_\omega = \{g \in G : g\omega = \omega\},$$
where $g\omega$ denotes the action of the group element g on the point ω.
   (a) Prove that $hG_\omega h^{-1} = G_{h\omega}$ for every $h \in G$.
   (b) Assume that G is abelian. Let N be the kernel of the transitive action. Prove that $N = G_\omega$ for every $\omega \in \Omega$.
   (c) Show that part (b) is not true if G is not abelian. In other words, give an example of a finite group G and a nonempty set Ω on which G acts transitively on the left such that $N \neq G_\omega$ for some ω.

2. Let N be a normal subgroup of the group G, and for each $g \in G$, let ϕ_g denote conjugation by g acting on N, i.e,
$$\phi_g(x) = gxg^{-1} \quad \text{for all } x \in N.$$
   (a) Prove that ϕ_g is an automorphism of N for each $g \in G$.
   (b) Prove that the map $\Phi\colon g \mapsto \phi_g$ is a homomorphism from G into $\mathrm{Aut}(N)$.
   (c) Prove that $\ker \Phi = C_G(N)$ and deduce that $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$.

3. Let G be a finite group acting transitively on the left on a nonempty set Ω. Let $N \trianglelefteq G$, and let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the orbits of N acting on Ω. For any $g \in G$, let
$$g\mathcal{O}_i = \{g\alpha : \alpha \in \mathcal{O}_i\}.$$
   (a) Prove that $g\mathcal{O}_i$ is an orbit of N for any $i \in \{1, 2, \ldots, r\}$, i.e., $g\mathcal{O}_i = \mathcal{O}_j$ for some j.
   (b) With G acting as in part (a), explain why G permutes $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ transitively.
   (c) Deduce from (b) that $r = [G : NG_\alpha]$, where G_α is the subgroup of G stabilizing the point $\alpha \in \mathcal{O}_1$.

4. (a) Find all finite groups G such that $\#\,\mathrm{Aut}(G) = 1$.
   (b) Argue that your argument from part (a) applies directly to infinite groups as well to find all infinite groups G with $\#\,\mathrm{Aut}(G) = 1$.