Math 395 - Fall 2021
Qual problem set 11

This homework is "due" on Monday November 29 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let K/F be a Galois extension of degree 4, where K and F are fields of characteristic different from 2. Show that $\mathrm{Gal}(K/F) \cong C_2 \times C_2$ if and only if there exist $x, y \in F$ such that $K = F(\sqrt{x}, \sqrt{y})$ and none of x, y or xy are squares in F.

2. Let K/F be an extension of odd degree, where F is any field of characteristic 0.
   (a) Let $\alpha \in F$ and assume the polynomial $x^2 - \alpha$ is irreducible over F. Prove that $x^2 - \alpha$ is also irreducible over K.
   (b) Assume further that K is Galois over F. Let $\alpha \in K$ and let E be the Galois closure of $K(\sqrt{\alpha})$ over F. Prove that $[E : F] = 2^r[K : F]$ for some $r \geq 0$.

3. Let p be a prime, let F be a field of characteristic 0, let E be the splitting field over F of an irreducible polynomial of degree p, and let $G = \mathrm{Gal}(E/F)$.
   (a) Explain why $[E : F] = pm$ for some integer m with $\gcd(p, m) = 1$.
   (b) Prove that if G has a normal subgroup of order m, then $[E : F] = p$ (i.e. $m = 1$).
   (c) Assume $p = 5$ and E is *not* solvable by radicals over F. Show that there are exactly 6 fields K with $F \subseteq K \subseteq E$ and $[E : K] = 5$.
   (You may quote without proof basic facts about groups of small order.)

4. Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of degree n and let K be the splitting field of $f(x)$ in $\mathbb{C}$. Assume that $G = \mathrm{Gal}(K/\mathbb{Q})$ is *abelian*.
   (a) Prove that $[K : \mathbb{Q}] = n$ and that $K = \mathbb{Q}(\alpha)$ for every root α of $f(x)$.
   (b) Prove that G acts regularly on the set of roots of $f(x)$. (A group acts regularly on a set if it is transitive and the stabilizer of any point is the identity.)
   (c) Prove that either all the roots of $f(x)$ are real numbers or none of its roots are real.
   (d) Is the converse of (a) true? That is, if K is the splitting field of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ and $\alpha \in K$ is a root of f such that $K = \mathbb{Q}(\alpha)$, must $\mathrm{Gal}(K/\mathbb{Q})$ be abelian?

5. Let F be a field of characteristic 0 and let $f \in F[x]$ be an irreducible polynomial of degree > 1 with splitting field $E \supset F$. Define $\Omega = \{\alpha \in E : f(\alpha) = 0\}$.

(a) Let $\alpha \in \Omega$ and let m be a positive integer. If $g \in F[x]$ is the minimal polynomial of α^m over F, show that $\{\beta^m : \beta \in \Omega\}$ is the set of roots of g.

(b) Now fix $\alpha \in \Omega$ and suppose that $\alpha r \in \Omega$ for some $r \in F$. Show that, for all $\beta \in \Omega$ and integers $i \geq 0$, we have $\beta r^i \in \Omega$. Conclude that r is a root of unity.

(c) If α and r are as in (b) and if m is the multiplicative order of the root of unity r, show that $f(x) = g(x^m)$, where g is the minimal polynomial of α^m over F.