Math 395 - Fall 2021
Qual problem set 10

This homework is "due" on Monday November 15 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let E be the splitting field in $\mathbb{C}$ of the polynomial $p(x) = x^6 + 3x^3 - 10$ over $\mathbb{Q}$, and let α be any root of $p(x)$ in E.
   (a) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. Be sure to justify your answer.
   (b) Describe the roots of $p(x)$ in terms of radicals involving rational numbers and roots of unity.
   (c) Find $[E : \mathbb{Q}]$. Be sure to justify your answer.
   (d) Prove that E contains a *unique* subfield F with $[F : \mathbb{Q}] = 2$.
2. Let $f(x) = x^6 - 6x^3 + 1$ and let α, β be the two real roots of f with $\alpha > \beta$. You may assume $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let K be the splitting field of $f(x)$ in $\mathbb{C}$.
   (a) Exhibit all six roots of $f(x)$ in terms of radicals involving only integers and powers of ω, where ω is a primitive cube root of unity.
   (b) Prove that $K = \mathbb{Q}(\alpha, \omega)$ and deduce that $[K : \mathbb{Q}] = 12$. (Hint: What is $\alpha\beta$?)
   (c) Prove that $G = \mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup N such that G/N is the Klein group of order four (this is $C_2 \times C_2$).
3. Let K be the splitting field of $(x^2 - 3)(x^3 - 5)$ over $\mathbb{Q}$.
   (a) Find the degree of K over $\mathbb{Q}$.
   (b) Find the isomorphism type of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
   (c) Find, with justification, all subfields F of K such that $[F : \mathbb{Q}] = 2$.
4. Let $f(x) = x^4 - 8x^2 - 1 \in \mathbb{Q}[x]$, let α be the real positive root of $f(x)$, let β be a nonreal root of $f(x)$ in $\mathbb{C}$, and let K be the splitting field of $f(x)$ in $\mathbb{C}$.
   (a) Describe α and β in terms of radicals involving integers, and deduce that $K = \mathbb{Q}(\alpha, \beta)$.
   (b) Show that $[\mathbb{Q}(\beta^2) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\beta) : \mathbb{Q}(\beta^2)] = 2$. Deduce from this that $f(x)$ is irreducible over $\mathbb{Q}$.
   (c) Show that $[K : \mathbb{Q}] = 8$ and that $\mathrm{Gal}(K/\mathbb{Q}) \cong D_4$.