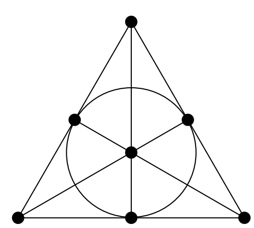
Math 395 - Fall 2020
Homework 3

This homework is due on Friday, September 18 to your peer reviewer, and on Friday, September 25 on Gradescope.

1. Let G be a finite group acting transitively on the left on a nonempty set Ω. Let $N \trianglelefteq G$, and let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the orbits of N acting on Ω. For any $g \in G$, let
$$g\mathcal{O}_i = \{g\alpha : \alpha \in \mathcal{O}_i\}.$$
   (a) Prove that $g\mathcal{O}_i$ is an orbit of N for any $i \in \{1, 2, \ldots, r\}$, i.e., $g\mathcal{O}_i = \mathcal{O}_j$ for some j.
   (b) With G acting as in part (a), explain why G permutes $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ transitively.
   (c) Deduce from (b) that $r = [G : NG_\alpha]$, where G_α is the subgroup of G stabilizing the point $\alpha \in \mathcal{O}_1$.

2. Let N be a normal subgroup of the group G, and for each $g \in G$, let ϕ_g denote conjugation by g acting on N, i.e,
$$\phi_g(x) = gxg^{-1} \quad \text{for all } x \in N.$$
   (a) Prove that ϕ_g is an automorphism of N for each $g \in G$.
   (b) Prove that the map $\Phi \colon g \mapsto \phi_g$ is a homomorphism from G into $\mathrm{Aut}(N)$.
   (c) Prove that $\ker \Phi = C_G(N)$ and deduce that $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$.

3. Consider the graph depicted below (where the vertices are the solid dots):



An *automorphism* of a graph is any permutation of vertices that sends edges to edges. Let G be the group of all automorphisms of this graph (the operation is composition).
   (a) Explain why G is isomorphic to a subgroup of S_7, and show that G has three orbits in this action.
   (b) Show that the order of G is not divisible by 5 or 7.
   (c) Prove that $G \cong D_3$.

4. Let $\mathcal{G}$ be a graph with a finite number of vertices and edges. Assume $\mathcal{G}$ has two connected components, $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $A = \mathrm{Aut}(\mathcal{G})$ be the group of all automorphisms of $\mathcal{G}$. (An *automorphism* of a graph is any permutation of vertices that sends edges to edges.)

   (a) Prove that if $\mathcal{G}_1$ has a different number of vertices from $\mathcal{G}_2$, then $A \cong \mathrm{Aut}(\mathcal{G}_1) \times \mathrm{Aut}(\mathcal{G}_2)$.

   (b) Give an example to show that the conclusion of (a) may be false if the hypothesis that $\mathcal{G}_1$ and $\mathcal{G}_2$ have different numbers of vertices is removed.

   (c) Describe why A must be solvable if both $\mathcal{G}_1$ and $\mathcal{G}_2$ each have at most 4 vertices.

5. (a) Find all finite groups G such that $\#\,\mathrm{Aut}(G) = 1$.

   (b) Argue that your argument from part (a) applies directly to infinite groups as well to find all infinite groups G with $\#\,\mathrm{Aut}(G) = 1$.

6. In this problem, G is a finite group.

   (a) Show that if $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G, then G is abelian.

   (b) Let p be a prime and P be a p-group. Show that $Z(P)$ is nontrivial.

   (c) Show that if P has order p^2 then P is abelian.

   (d) Show that every p-group is solvable.