

COMPLEX ANALYSIS

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Last week: power series for holomorphic functions

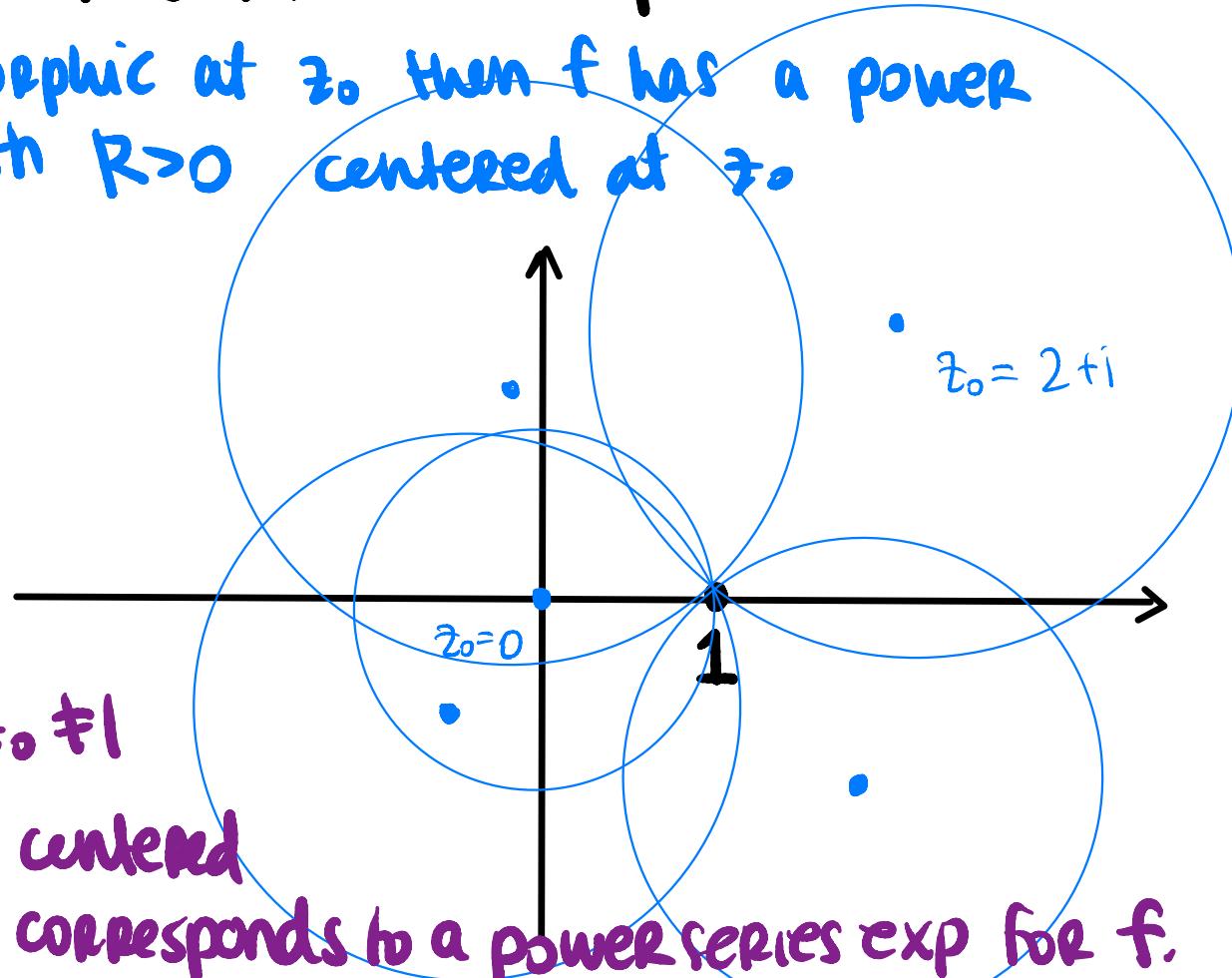
if f is holomorphic at z_0 then f has a power series with $R > 0$ centered at z_0

$$f(z) = \frac{-z}{1-z}$$

holomorphic on

$$U = \mathbb{C} - \{1\}$$

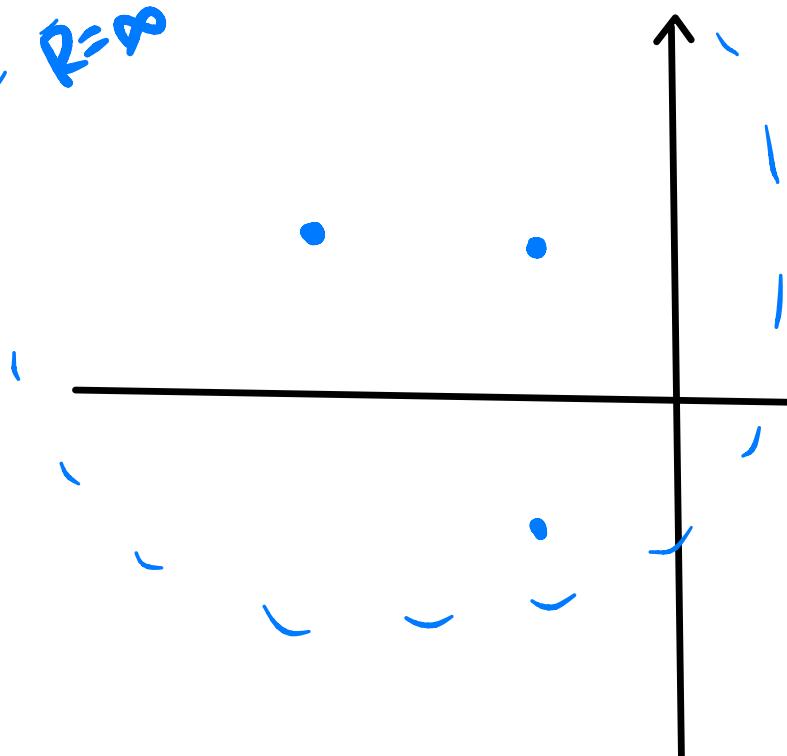
By moving around $z_0 \neq 1$
cover \mathbb{C} with circles centered
at z_0 and each circle corresponds to a power series exp for f .



$$f(z) = \exp(z) = \exp(z_0) \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} \quad R=\infty$$

$U = \mathbb{C}$

$R=\infty$



$$\bullet z_0 = 3+i \quad \exp(z) = \exp(3+i) \sum_{k=0}^{\infty} \frac{(z-3-i)^k}{k!}$$

$$\bullet z_0 = 2-i$$

This week: What if we need to stick to a fixed z_0 ?

$$f(z) = \frac{-z}{1-z}$$

$$z_0 = 0$$

$$\begin{aligned} f(z) &= -z \cdot \frac{1}{1-z} \\ &= -z \cdot \sum_{k=0}^{\infty} z^k = -\sum_{k=0}^{\infty} z^{k+1} \quad R=1 \end{aligned}$$

f is holomorphic
out here

$$f(z) = \sum_{k=0}^{\infty} z^{-k}$$

$$\begin{aligned} |z| > 1 \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \dots \end{aligned}$$

$$f(z) = -\sum_{k=1}^{\infty} z^k$$

$$|z| < 1 \\ = -z - z^2 - z^3 - \dots$$

← these 2
series give you
f everywhere except
maybe $|z|=1$

Section 8.3 of BMPS : Laurent series

Definition: A Laurent series centered at $z_0 \in \mathbb{C}$ is a double series of the form

$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k := \sum_{k=0}^{\infty} c_k (z - z_0)^k + \sum_{k=1}^{\infty} c_{-k} (z - z_0)^{-k}$$

||

The diagram illustrates the decomposition of a Laurent series. A blue wavy line starts from the left and points to the first term $c_{-2}(z-z_0)^{-2}$. From there, it continues straight to the term $c_{-1}(z-z_0)^{-1}$, which is highlighted with a green bracket below. From $c_{-1}(z-z_0)^{-1}$, a blue arrow points right to the term $c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots$. This entire right-hand side is also highlighted with a green bracket below. Above the terms, a green arrow points from the positive k sum to the negative k sum, indicating they are added together.

$$\dots + c_{-2} (z - z_0)^{-2} + c_{-1} (z - z_0)^{-1} + c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \dots$$

$\underbrace{\dots}_{k=2} \quad \underbrace{k=1 \quad \quad \quad k=0 \quad k=1 \quad \quad \quad k=2}_{\dots}$

The double series converges iff both series converge.
if and only if

We know that the "power series" half converges in a circle around z_0 .

$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k := \sum_{k=0}^{\infty} c_k (z - z_0)^k + \sum_{k=1}^{\infty} c_{-k} (z - z_0)^{-k}$$

$\exists R$ s.t. this converges for $|z - z_0| < R$

$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k := \sum_{k=0}^{\infty} c_k (z - z_0)^k + \sum_{k=1}^{\infty} c_{-k} (z - z_0)^{-k}$$

$= a_k$

Where does the other half converge?

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k \rightarrow \infty} \frac{|c_{-(k+1)} (z - z_0)^{-(k+1)}|}{|c_{-k} (z - z_0)^{-k}|} \\ &= \lim_{k \rightarrow \infty} \left| \frac{c_{-(k+1)}}{c_{-k}} \right| \frac{1}{|z - z_0|} \\ &= \frac{1}{|z - z_0|} \lim_{k \rightarrow \infty} \left| \frac{c_{-(k+1)}}{c_{-k}} \right| \end{aligned}$$

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{|z - z_0|} \lim_{k \rightarrow \infty} \frac{|c - (z_{k+1})|}{|c - z_k|}$$

L

if this limit doesn't exist then whole limit and series diverges

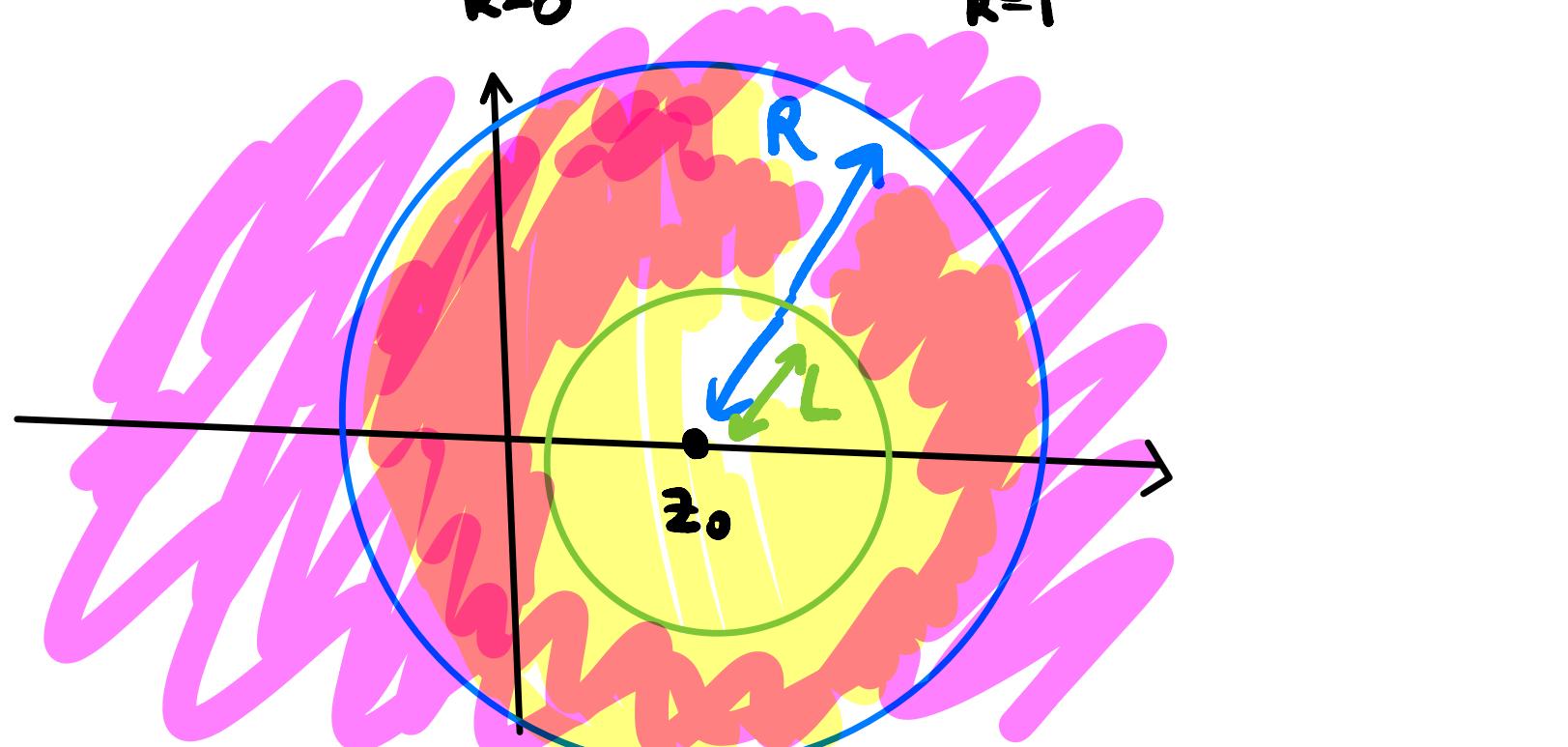
Ratio test says my series converges

if $\frac{1}{|z - z_0|} L < 1 \Rightarrow L < |z - z_0|$

my series diverges if

$$\frac{1}{|z - z_0|} L > 1 \Rightarrow L > |z - z_0|$$

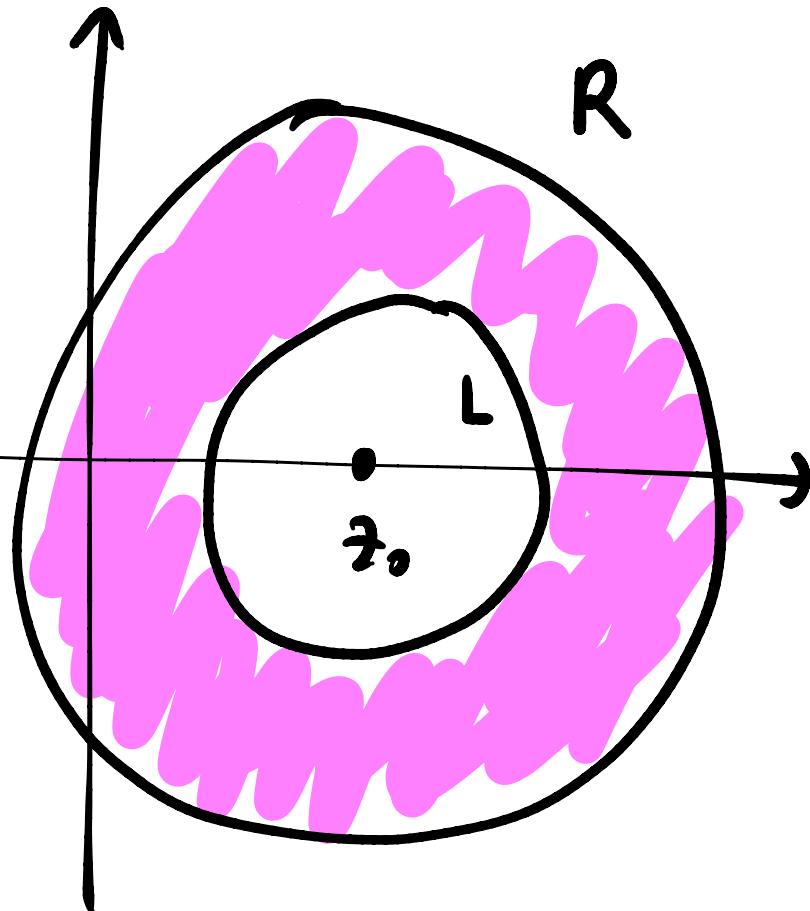
$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k := \underbrace{\sum_{k=0}^{\infty} c_k (z - z_0)^k}_{R \quad |z - z_0| < R} + \underbrace{\sum_{k=1}^{\infty} c_{-k} (z - z_0)^{-k}}_{L \quad |z - z_0| > L}$$



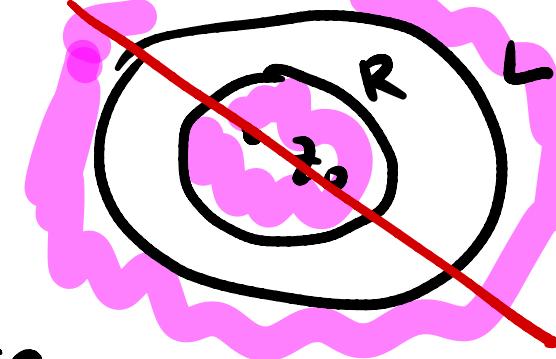
IF $L < R$

the Laurent series
converges in the
annulus

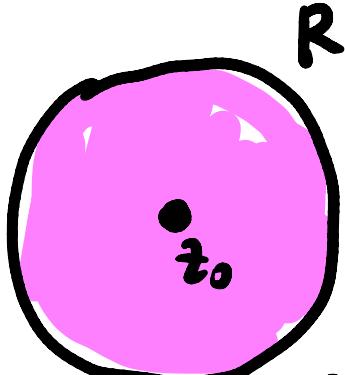
$$L < |z - z_0| < R$$



Special cases: if $R < L$
no intersection where both
converge so Laurent
series does not converge
anywhere



if $L=0$



series converges for $0 < |z - z_0| < R$

The Laurent series converges
at $z = z_0$ iff $c_k = 0$ for $k < 0$
(this is the case where the
Laurent series is a
power series)

if $R=\infty$, series converges for $|z-z_0| < L$



There will be
warm up 9.2

THAT'S ALL FOR TODAY!