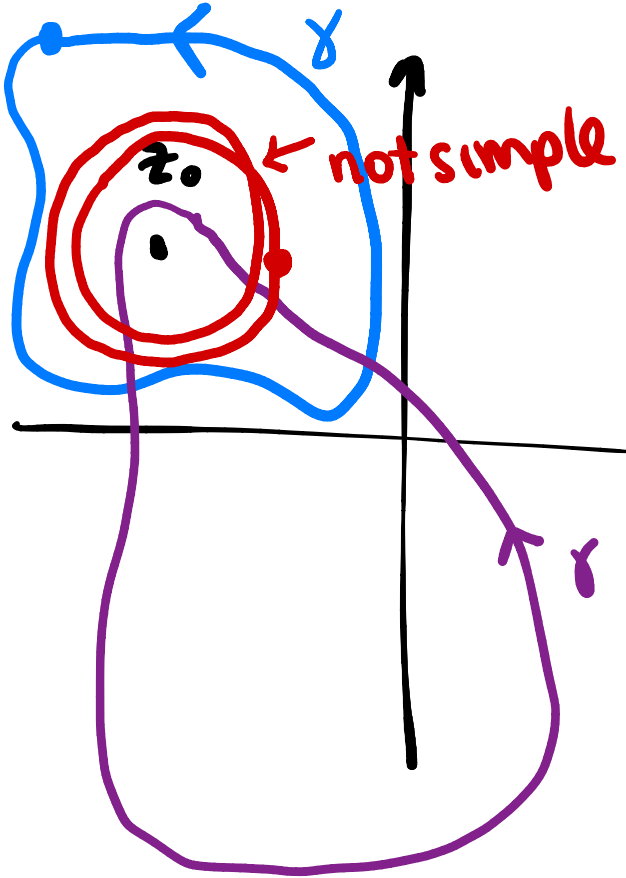
Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

- There will be email when HW is graded.
- Warm up 8.1 by the end of today

#1a) $z_0 \in \mathbb{C}$, γ positively oriented



simple (doesn't cross itself except maybe beginning & end)

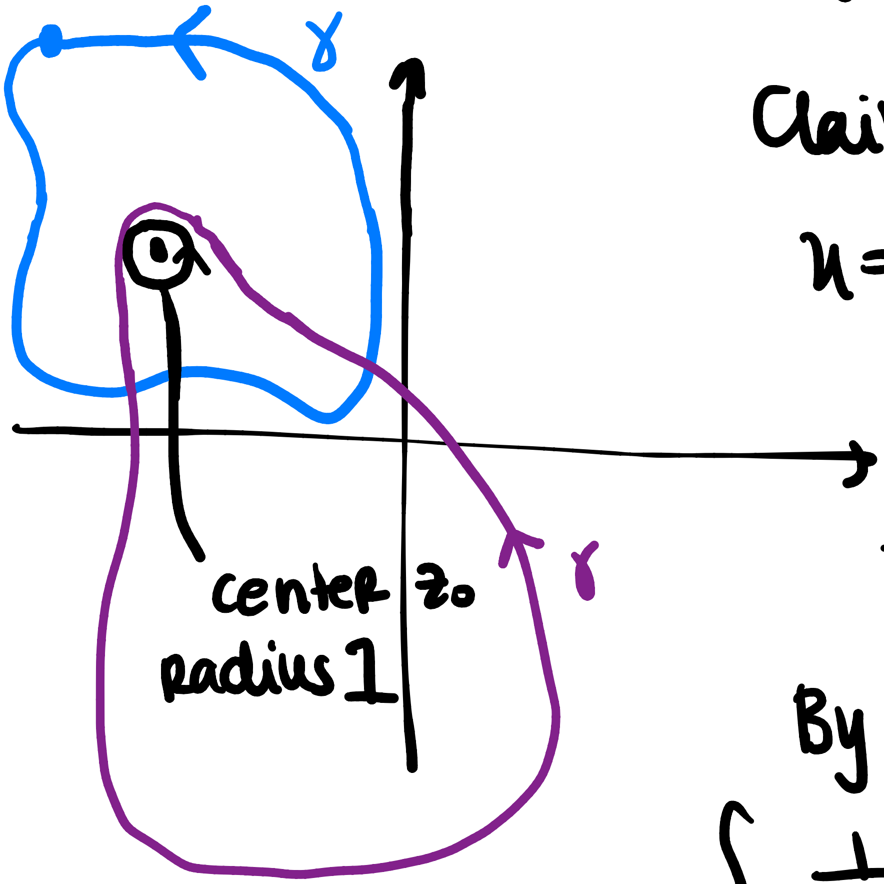
closed

(does begin + end in same place)

not simple

blue + purple are 2 possibilities

$\Rightarrow \gamma$ goes around z_0 once.



z_0 is an interior point of γ

Claim: $\gamma \sim_{\mathcal{U}}$ circle around z_0 of radius 1

$\mathcal{U} = \mathbb{C} - \{z_0\}$ open connected

granted

$f(z) = \frac{1}{z - z_0}$ holomorphic in $\mathcal{U}$

By Cauchy's theorem

$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\text{circle}} \frac{1}{z-z_0}\,dz$$

$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\gamma_1}\frac{1}{z-z_0}\,dz$$

where γ_1 is the circle of radius 1 around z_0

$\gamma_1(t) = z_0 + e^{it}$

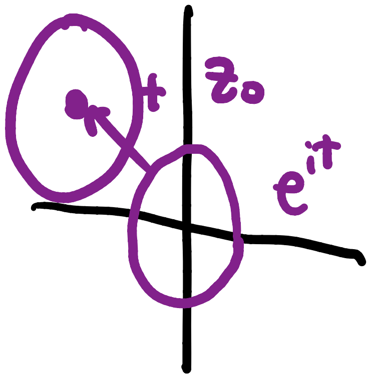
$0 \le t \le 2\pi$

$$= \int_0^{2\pi} \overbrace{\frac{1}{(\cancel{z_0}+e^{it})-\cancel{z_0}}}^{f(\gamma_1(t))}\ \overbrace{ie^{it}\,dt}^{\gamma_1'(t)\,dt}$$

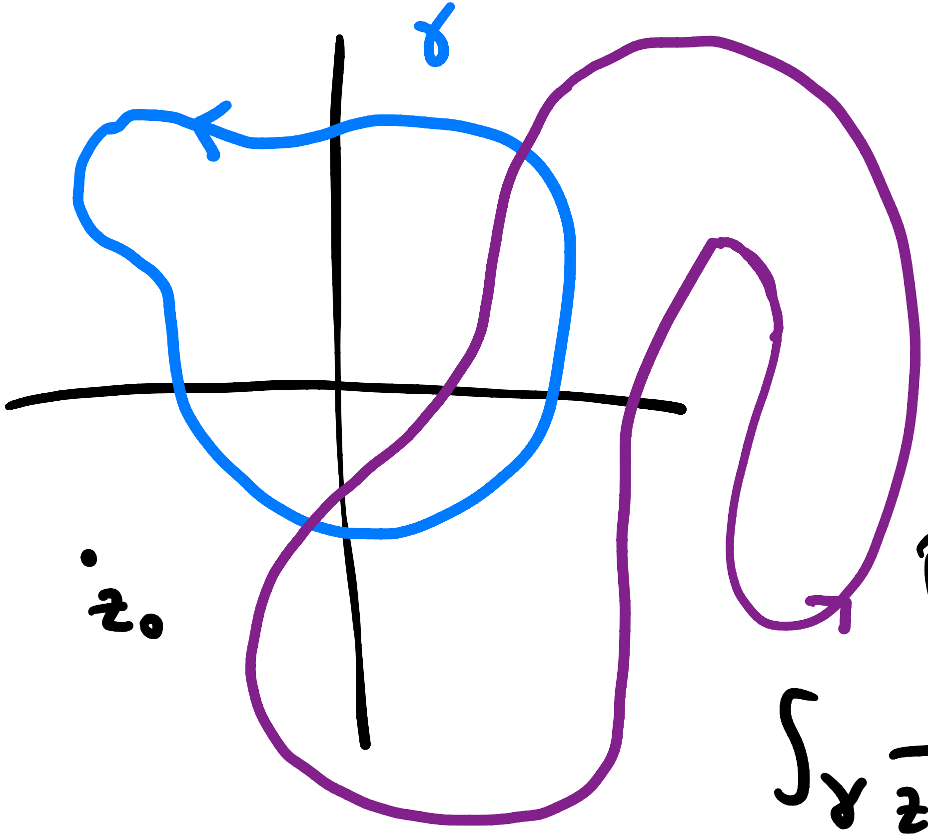
$\gamma_1'(t) = 0 + ie^{it}$

$$= \int_0^{2\pi} \frac{i\cancel{e^{it}}}{\cancel{e^{it}}}\,dt$$

$$= \int_0^{2\pi} i\,dt = it\Big|_0^{2\pi} = i(2\pi-0) = 2\pi i$$



What if z_0 is an exterior point?



Claim: $\gamma \sim_U 0$

constant path, stay in one spot

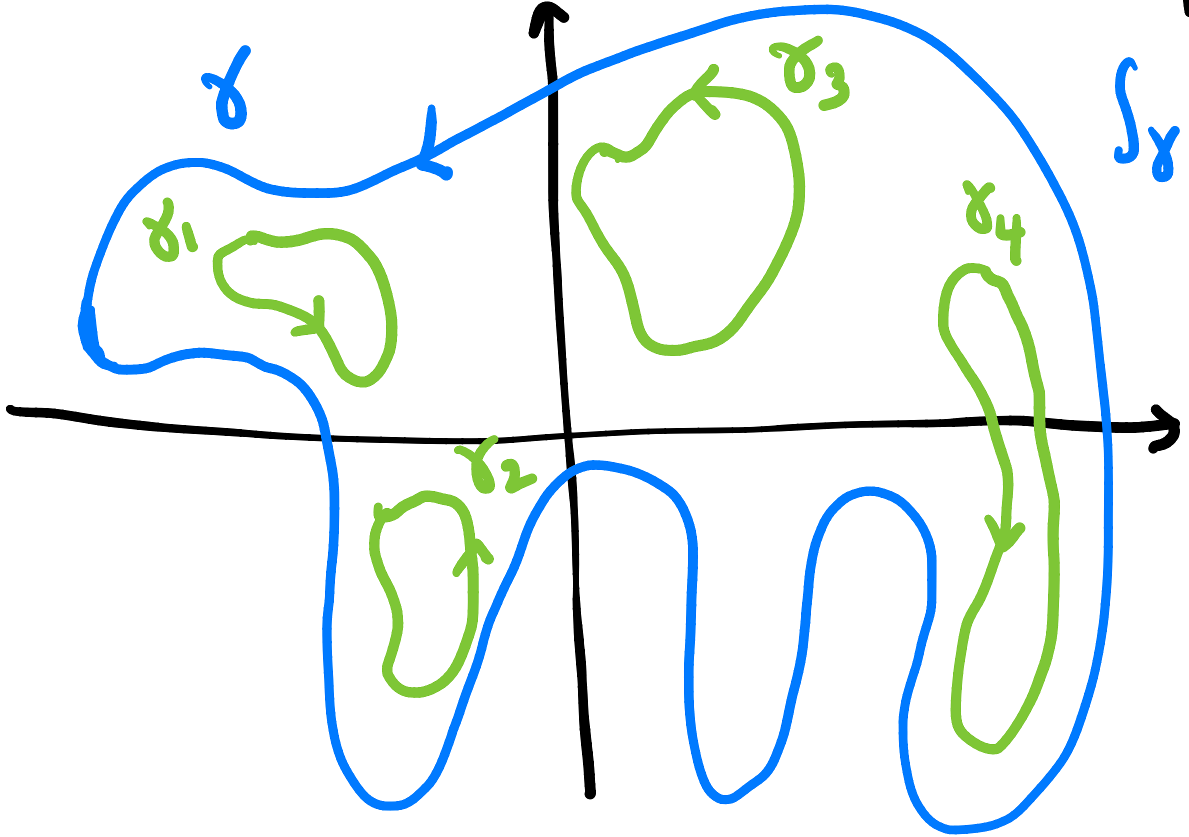
$U = \mathbb{C} - \{z_0\}$

$f(z) = \frac{1}{z - z_0}$ is holomorphic in U

By Cauchy's Thm

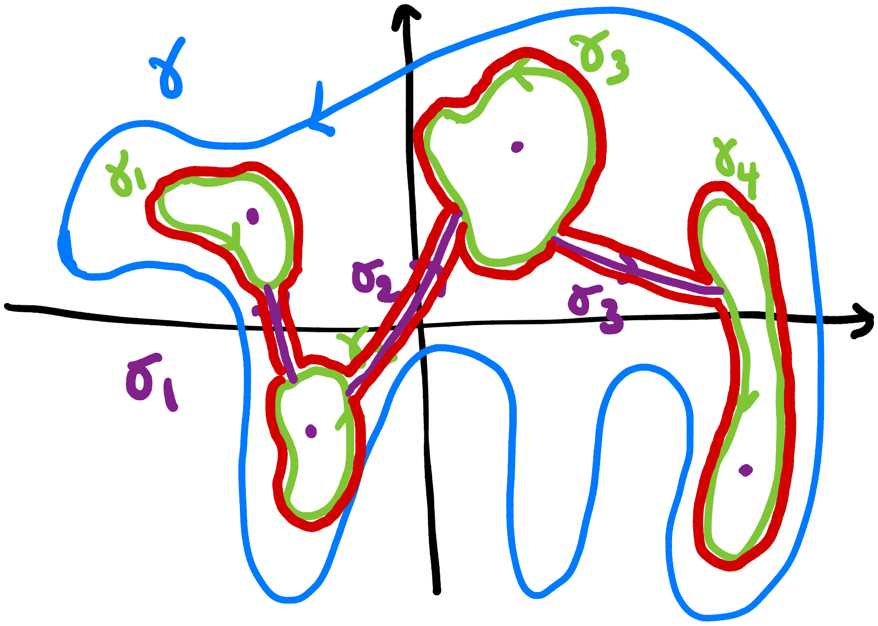
$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\text{constant path}} \frac{1}{z-z_0}\,dz = 0$$

#1b) f entire function: f is holomorphic on $U = \mathbb{C}$



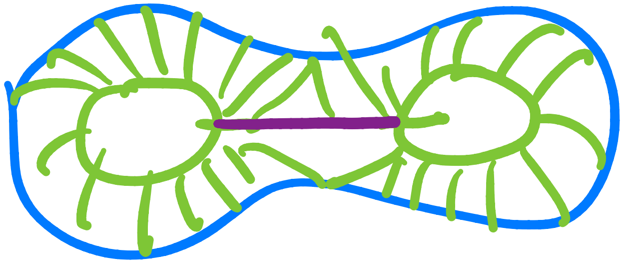
To show

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$



Claim:

$$\gamma \sim_{\mathbb{C}} \sigma = \gamma_1 + \sigma_1 + \text{part of } \gamma_2 + \sigma_2 + \text{part of } \gamma_3 + \sigma_3 + \gamma_4 - \sigma_3 + \text{rest of } \gamma_3 - \sigma_2 + \text{rest of } \gamma_2 - \sigma_1$$



$$
\begin{aligned}
\int_{\gamma} f(z)\,dz = {} & \int_{\gamma_1} f(z)\,dz + \cancel{\int_{\sigma_1} f(z)\,dz} + \int_{\gamma_2} f(z)\,dz \\
& \cancel{\int_{\sigma_2} f(z)\,dz} + \int_{\gamma_3} f(z)\,dz + \cancel{\int_{\sigma_3} f(z)\,dz} \\
& + \int_{\gamma_4} f(z)\,dz - \cancel{\int_{\sigma_3} f(z)\,dz} \\
& - \cancel{\int_{\sigma_2} f(z)\,dz} - \cancel{\int_{\sigma_1} f(z)\,dz}
\end{aligned}
$$

Note that since f is entire

$$\int_{\gamma} f(z)\,dz = 0$$

$$\int_{\gamma_i} f(z)\,dz = 0$$

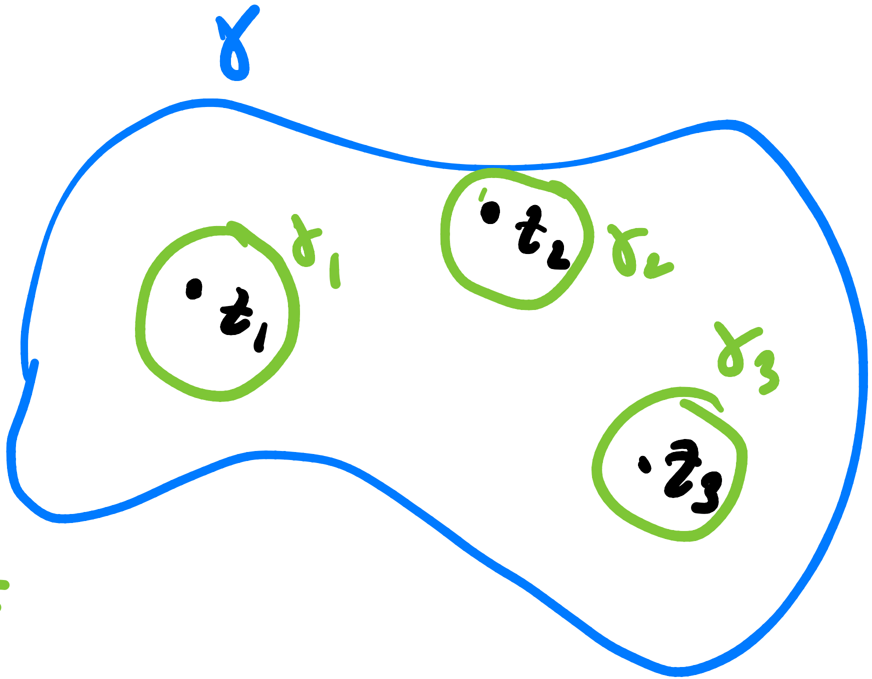
$$0 = 0 + 0 + 0 + 0$$

$$\gamma \sim_{\mathbb{C}} \gamma_i$$

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_i} f(z)\,dz = 0$$

Interesting application is when there are missing points inside the γ_i's

$$\mathbb{C} - \{z_1, z_2, z_3\}$$



$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$

I will record myself doing

1c)

2b)

and post with our class videos ~ 1pm

That's all for today!