Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Week 7 is starting to go up on the website

2 warm ups

videos + reading up

HW7 is not up – will be by end of day Saturday

Today mostly do HW6

#1b) $\int_\gamma x\,dz$ $\quad x = \mathrm{Re}(z)$ $\quad \gamma(t) = e^{it}$ $\quad 0 \le t \le 2\pi$

$$= \int_\gamma \mathrm{Re}(z)\,dz$$

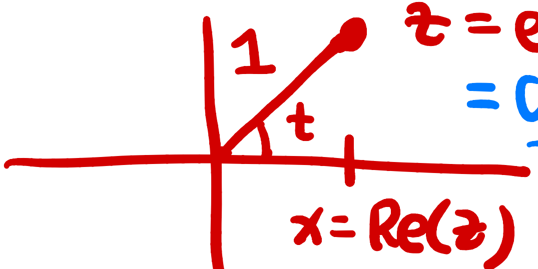
$$= \int_0^{2\pi} \underbrace{\cos t}_{\mathrm{Re}(e^{it})} \; \underbrace{i e^{it}}_{\gamma'(t)} \, dt$$

$$= i \int_0^{2\pi} \cos t \, e^{it} \, dt$$

$$\boxed{\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t))\,\gamma'(t)\,dt}$$

$$z = \gamma(t) = e^{it}$$

$$\mathrm{Re}(z) = \mathrm{Re}(e^{it})$$



$z = e^{it}$

$= \underline{\cos t} + i\sin t$

$$\int_\gamma \mathrm{Re}(z)\,dz = i\int_0^{2\pi} \cos t \underbrace{e^{it}}_{\cos t + i\sin t} dt$$

$$= i\int_0^{2\pi} \cos t(\cos t + i\sin t)\,dt$$

$$= i\int_0^{2\pi} (\cos^2 t + i\sin t\cos t)\,dt$$

$$= i\int_0^{2\pi} \cos^2 t\,dt - \int_0^{2\pi} \sin t\cos t\,dt$$

1st thought
integration by parts

$$\int u\,dv = uv - \int v\,du$$

$u = \cos t \qquad dv = e^{it}dt$

$du = -\sin t\,dt \qquad v = \frac{e^{it}}{i}$

$= \ldots + i\int \sin t\, e^{it}\,dt$

$u = \sin t \qquad dv = e^{it}dt$

$du = \cos t\,dt \qquad v = \frac{e^{it}}{i}$

$= i\int \cos t\, e^{it}\,dt$

$$\int_\gamma \mathrm{Re}(z)\,dz = i\int_0^{2\pi}\cos^2 t\,dt - \int_0^{2\pi}\sin t\cos t\,dt$$

$u = \sin t$
$du = \cos t\,dt$

$t = 0$
$u = \sin 0 = 0$
$t = 2\pi$
$u = \sin 2\pi = 0$

$$= i\int_0^{2\pi}\frac{1}{2}(1+\cos 2t)\,dt - \int_0^0 u\,du$$

$u = 2t$
$du = 2dt$
$\frac{1}{2}du = dt$

$$= \frac{i}{2}\int_0^{2\pi}(1+\cos 2t)\,dt - \left.\frac{u^2}{2}\right|_0^0$$

$$= \frac{i}{4}\int_0^{4\pi}(1+\cos u)\,du - 0$$

$$= \frac{i}{4}(u+\sin u\Big|_0^{4\pi} - 0$$

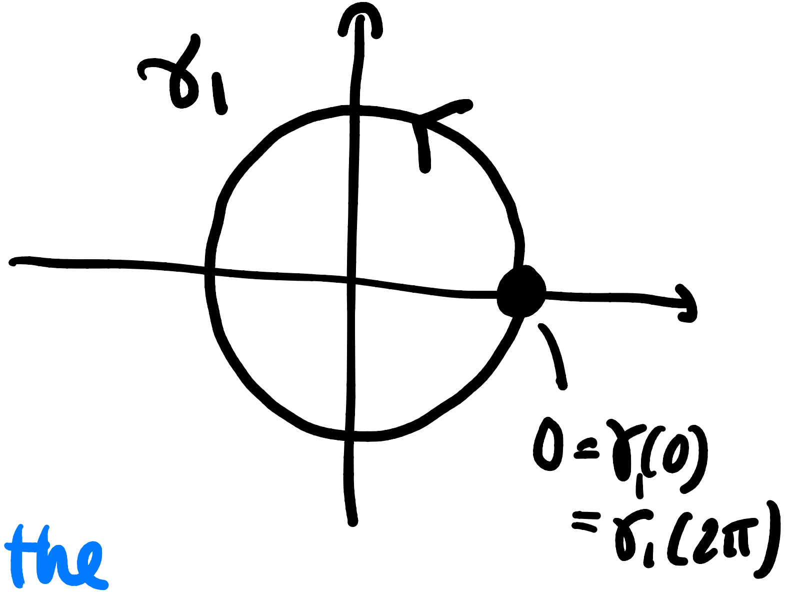
$$= \frac{i}{4}(4\pi+0-0-0) - 0$$

$$= \pi i$$

In class at some point:

$$\int_{\gamma_1} \frac{1}{z}\, dz = 2\pi i$$

$\gamma_1(t) = e^{it} \quad 0 \leq t \leq 2\pi$



Note that γ_1 starts + ends in the same spot so <u>if</u> $\frac{1}{z}$ had an antiderivative F we would have

end point start point

$$\int_{\gamma_1} \frac{1}{z}\, dz = F(0) - F(0) = 0$$

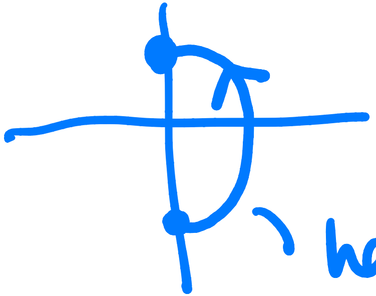
Because $\int_{\gamma_1} \frac{1}{z} dz \neq 0$, we can say that $\frac{1}{z}$ does not have an antiderivative on an open set U containing γ_1

But in HW6 asked to do $\int_{\gamma_2} \frac{1}{z} dz$

$\gamma_2(t) = e^{it}$

$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

using an antiderivative!!



half of unit circle

Answer is that $\frac{1}{z}$ can have an antiderivative but never on all of $\mathbb{C}$!

Because we require that the antiderivative be <u>holomorphic</u>!

F is an antiderivative for f on U (open set) if $F'(z) = f(z) \quad \forall z \in U$

↳ F is holomorphic on U!

An antiderivative for $\frac{1}{z}$ is any branch of $\log z$, which is always holomorphic on

$$\mathbb{C} - \{\text{a ray}\}$$

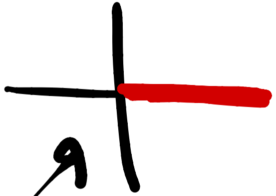
$$\underbrace{\log z}_{\text{multivalued function}} = \ln|z| + i\,\underbrace{\arg(z)}_{\text{single-valued by picking an argument}}$$

Examples of branches of $\arg(z)$:

Principal branch $\mathrm{Arg}(z) \in (-\pi, \pi]$

discontinuity

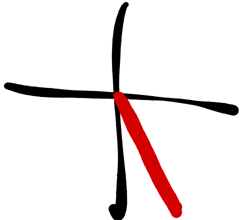
Christelle branch $\arg_c(z) \in [0, 2\pi)$



$\log_c(z) = \ln|z| + i\arg_c(z)$

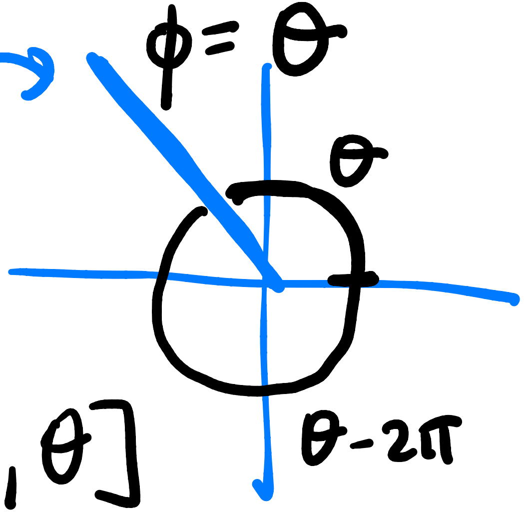
is holomorphic on $\mathbb{C} - \{\text{pos real axis}\}$

$\arg_w(z) \in \left(-\frac{\pi}{3}, \frac{5\pi}{3}\right]$



So you can have a holomorphic log function in $\mathbb{C} - \{\text{ray}\}$ by choosing your arg carefully

ok with missing this



choose $\arg(z) \in (\theta - 2\pi, \theta]$

How is that ok?

Notice that if $\log_1(z)$ and $\log_2(z)$ are 2 branches of $\log(z)$ then

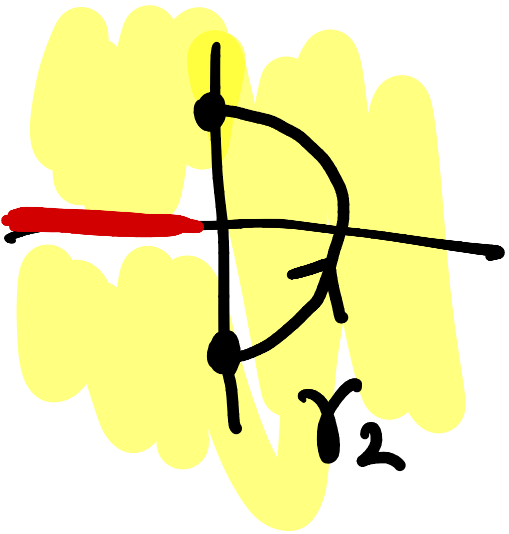
$$\log_1(z) = \log_2(z) + \underbrace{2\pi i k}_{\text{Constant}} \quad k \in \mathbb{Z}$$

from calculus

$$\int f\,dx = F(x) + C$$

HW6 #2c)

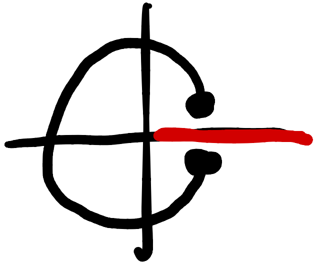
$$\int_{\gamma_2} \frac{1}{z}\,dz = \mathrm{Log}\left(\gamma_2\left(\tfrac{\pi}{2}\right)\right) - \mathrm{Log}\left(\gamma_2\left(-\tfrac{\pi}{2}\right)\right)$$

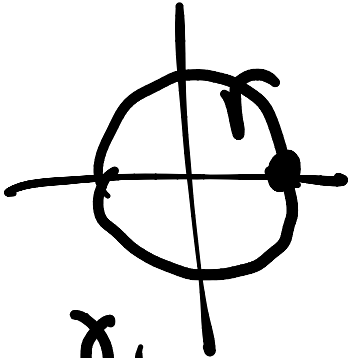


all we need is a log that is holomorphic in an open set containing γ_2

$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$

Going back to γ_1





γ_1

no branch of log holomorphic on an open set containing γ_1 because γ_1 intersects every ray in $\mathbb{C}$!

Aside on addition formulas for trig

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad x, y \in \mathbb{R}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$x = y = t$

$$\sin^2 t + \cos^2 t = 1$$

$$\cos^2 t - 1 = -\sin^2 t$$

$$\cos(2t) = \cos^2 t - \sin^2 t$$

$$= \cos^2 t + \cos^2 t - 1 = 2\cos^2 t - 1$$

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$

That's all for today!