Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Check in

Any questions or concerns? Anything unclear?

HW3 Problem 2

$$f(z) = \overbrace{f(x,y)}^{\in \mathbb{C}} = \boxed{\underbrace{\overbrace{u(x,y)}^{\in \mathbb{R}}}_{Re(f(z))} + i\underbrace{\overbrace{v(x,y)}^{\in \mathbb{R}}}_{Im(f(z))}} \in \mathbb{C}$$

$$z = x + iy$$

$$u(x,y) \rightsquigarrow \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$\forall z$ in the domain of f

$\nabla u \cdot \nabla v = 0$ i.e. ∇u, ∇v are perpendicular

dot product normal mult.

$$\parallel \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}$$

$\vec{v}_1 = (a_1, a_2)$

$\vec{v}_1 \cdot \vec{v}_2$ $\vec{v}_2 = (b_1, b_2)$

$\searrow$ $a_1 b_1 + a_2 b_2$

Region = open + connected

domain of a function = set of allowable inputs

Example from calculus:

domain $f(x) = \frac{1}{x}$? $(-\infty, 0) \cup (0, \infty)$

or $\mathbb{R} - \{0\}$

$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

C-R

$$= \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} + \left(-\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}$$

Goal $\rightarrow = 0$

f is holomorphic so u and v satisfy the C-R equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Show that $\nabla u = 0$ iff $\nabla v = 0$

Suppose that $\nabla u = 0$

This means that $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$

or $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

Then by C-R, $\frac{\partial v}{\partial y} = 0$ and $-\frac{\partial v}{\partial x} = 0$

C-R

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If $\nabla u = 0$ then $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0$

so $\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = (0,0) = 0$

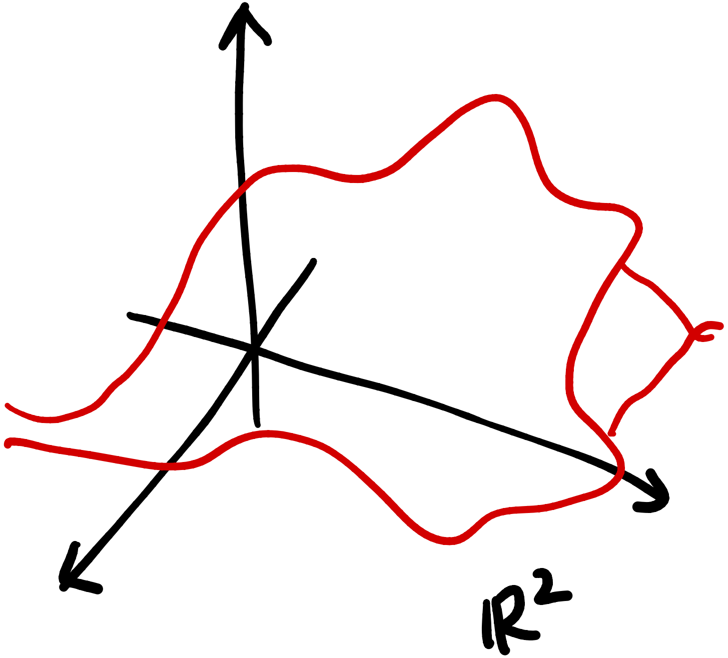
We showed that $\nabla u = 0 \Rightarrow \nabla v = 0$

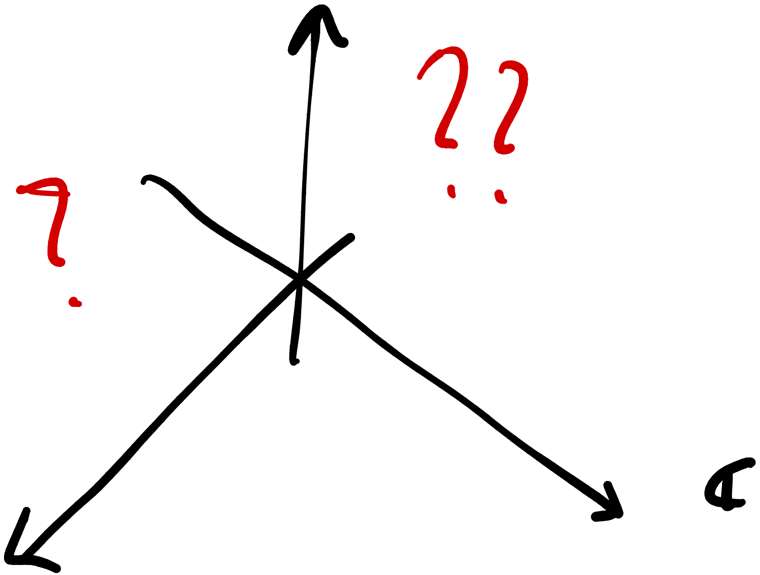
Let you try $\nabla v = 0 \Rightarrow \nabla u = 0$

<u>or</u> $\nabla u = 0 \Leftrightarrow \nabla v = 0$

In calc 3 $f: \mathbb{R}^2 \to \mathbb{R}$

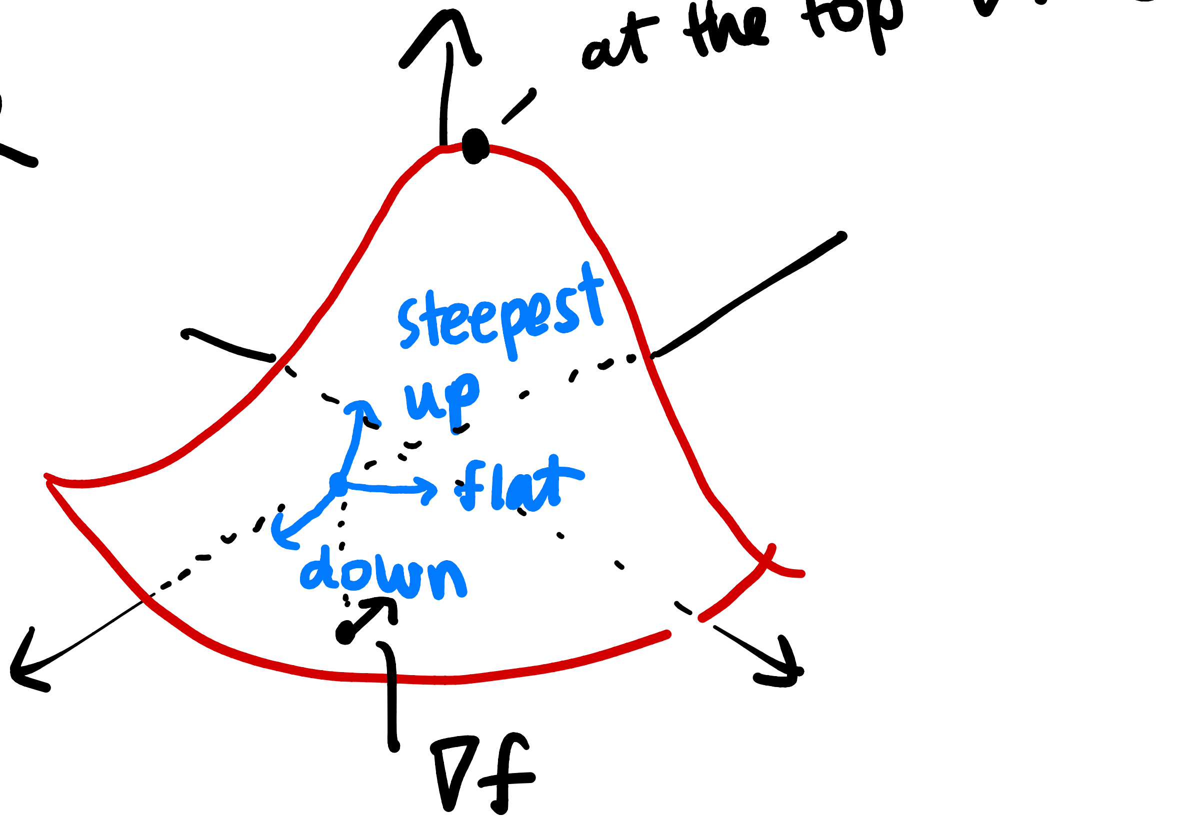
$$\begin{array}{ccccc} f: & \mathbb{C} & \to & \mathbb{C} \\ & \wr\wr & & \wr\wr \\ & \mathbb{R}^2 & & \mathbb{R}^2 \end{array}$$





Gradient of u is the direction in which u grows the fastest

$f: \mathbb{R}^2 \to \mathbb{R}$



OUR GOAL TODAY

Last class $f(z) = z^3 - 3z$ entire

holomorphic on all of $\mathbb{C}$

If f is a polynomial in z, then f is entire.

What if f is a polynomial in x (the real part of z) and y (the imaginary part of z)? Is f entire then? **Can** f be entire then? Under which circumstances?

Big question

$$f(z) = 2x^2 + iy - (1+2i)xy$$

entire?

Strategies to solve hard problems

- Don’t panic.
- If you have strong emotions, observe them and let yourself feel them.
- Write down theorems and definitions you might need.
- Try to think of an easier problem that you might be able to solve and which would give you insight.
- Sleep on it.
- Talk to others, even people who don’t study complex analysis!

Warm up problem

1. Suppose that $u(x, y) = x^2 - y^2$. Is u the real part of an entire function? If so, give v such that f = u + i v is entire.
2. Same question, but $u(x, y) = 2x^2 + x + 1 - 2y^2$.

Back to our big question

If f is a polynomial in z, then f is entire.

What if f is a polynomial in x (the real part of z) and y (the imaginary part of z)? Is f entire then? **Can** f be entire then? Under which circumstances?

That's all for today!