Two spaces associated to a matrix

Let $A \in M_{n \times m}$ have m columns $\vec{c}_1, \vec{c}_2 \ldots \vec{c}_m$. There are 2 spaces associated to A:

1 The null space

- This is a subspace of $\mathbb{R}^m$
- This is all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ such that

$$x_1 \vec{c}_1 + x_2 \vec{c}_2 + x_3 \vec{c}_3 + \ldots + x_m \vec{c}_m = \vec{0}$$

- We have seen this before as the space of solutions to the homogeneous system

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1m} x_m &= 0 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2m} x_m &= 0 \\ &\vdots \\ a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nm} x_m &= 0 \end{aligned}$$

2 The column space

- This is a subspace of $\mathbb{R}^n$
- This is all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ inside

$$\text{Span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_m\}$$

In other words, this is all $\vec{x}$ such that there are $a_1, a_2 \dots a_m$ with

$$a_1\vec{c}_1 + a_2\vec{c}_2 + \dots + a_m\vec{c}_m = \vec{x}$$

You are responsible for being able to compute a basis for both of these spaces.

Example: $A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix} \in M_{3\times 4}$

Question 1: Give a basis for the subspace of vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ such that

$$\begin{aligned} x_1 + 3x_2 - x_3 + 2x_4 &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 0 \\ x_2 + x_3 + 4x_4 &= 0 \end{aligned}$$

Another way to phrase this:

vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$

Solution: We are not given a spanning set so we

- solve
- give answer in vector form
- argue the vectors give a basis

Solve:
$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix} \overset{\rho_2 - 2\rho_1}{\sim} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -5 & 3 & -4 \\ 0 & 1 & 1 & 4 \end{pmatrix}$$

$$\overset{\rho_2 \leftrightarrow \rho_3}{\sim} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -5 & 3 & -4 \end{pmatrix} \overset{\rho_3 + 5\rho_2}{\sim} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 8 & 16 \end{pmatrix}$$

This is echelon form. Because I am solving I choose to continue to <u>reduced</u> echelon form.

$$\overset{\frac{1}{8}\rho_3}{\sim} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \underset{\rho_2 - \rho_3}{\overset{\rho_1 + \rho_3}{\sim}} \begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\overset{\rho_1 - 3\rho_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

x_4 is free

$x_1 = 2x_4$

$x_2 = -2x_4$

$x_3 = -2x_4$

so $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \\ 1 \end{pmatrix} x_4$$

basis is $\left\{ \begin{pmatrix} 2 \\ -2 \\ -2 \\ 1 \end{pmatrix} \right\}$ This is linearly independent because it is a single non zero vector.

The solution set is a subset of $\mathbb{R}^4$.

Question 2: Give a basis for the column space of A

Solution: We <u>are</u> given a spanning set, so we shrink it to a basis

- get matrix in echelon form
- basis is columns corresponding to leading variables

We already know an echelon form for A:

$$A \sim \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 8 & 16 \end{pmatrix}$$

$\uparrow \uparrow \uparrow$

leading variables

basis is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Our method guarantees this is a basis.

The column space is a subset of $\mathbb{R}^3$

This agrees with Theorem 3.13 :

A has $m=4$ columns

A has rank 3

The solution space of $A\vec{x}=\vec{0}$ has dimension 1

and $4 = 3 + 1$