One.III Reduced Echelon Form

Linear Algebra
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Gauss-Jordan reduction

Pivoting

Here is an extension of Gauss's Method with some advantages.

Example Start as usual with elimination operations to get echelon form.

$$\begin{array}{rcl} x + y - z = 2 \\ 2x - y \phantom{{}+2z} = -1 \\ x - 2y + 2z = -1 \end{array} \xrightarrow[-1\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} x + y - z = 2 \\ -3y + 2z = -5 \\ -3y + 3z = -3 \end{array}$$

$$\xrightarrow{-1\rho_2+\rho_3} \begin{array}{rcl} x + y - z = 2 \\ -3y + 2z = -5 \\ z = 2 \end{array}$$

Now, instead of doing back substitution, we continue using row operations. First make all the leading entries one.

$$\xrightarrow{(-1/3)\rho_2} \begin{array}{rcl} x + y - z = 2 \\ y - (2/3)z = 5/3 \\ z = 2 \end{array}$$

Finish by using the leading entries to eliminate upwards, until we can read off the solution.

$$\begin{aligned} x + y - z &= 2 \\ y - (2/3)z &= 5/3 \\ z &= 2 \end{aligned} \quad \xrightarrow[(2/3)\rho_3+\rho_2]{\rho_3+\rho_1} \quad \begin{aligned} x + y &= 4 \\ y &= 3 \\ z &= 2 \end{aligned}$$

$$\xrightarrow{-\rho_2+\rho_1} \quad \begin{aligned} x &= 1 \\ y &= 3 \\ z &= 2 \end{aligned}$$

Using one entry to clear out the rest of a column is *pivoting* on that entry.

Example With this system

$$\begin{aligned} x - y - 2w &= 2 \\ x + y + 3z + w &= 1 \\ -y + z - w &= 0 \end{aligned}$$

we can rewrite in matrix notation and do Gauss's Method.

$$\xrightarrow{-1\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & 0 & 5/2 & 1/2 & -1/2 \end{array}\right)$$

We can combine the operations turning the leading entries to 1.

$$\xrightarrow[(2/5)\rho_3]{(1/2)\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 3/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

Now eliminate upwards.

$$\xrightarrow{-(3/2)\rho_3+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right) \xrightarrow{\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

The final augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

gives the parametrized description of the solution set.

$$\{\begin{pmatrix} 9/5 \\ -1/5 \\ -1/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4/5 \\ -6/5 \\ -1/5 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R}\}$$

Gauss-Jordan reduction

This extension of Gauss's Method is the *Gauss-Jordan Method* or *Gauss-Jordan reduction*.

1.3 *Definition* A matrix or linear system is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a 1 and is the only nonzero entry in its column.

The cost of using Gauss-Jordan reduction to solve a system is the additional arithmetic. The benefit is that we can just read off the solution set description.