Describing the solution set

Parametrizing

We've seen that this system has infinitely many solutions.

$$\begin{array}{rcl} -x - y + 3z = 3 \\ x \quad + z = 3 \\ 3x - y + 7z = 15 \end{array} \xrightarrow[3\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-4\rho_2+\rho_3} \begin{array}{r} -x - y + 3z = 3 \\ -y + 4z = 6 \\ 0 = 0 \end{array}$$

We want to describe the solution set.

Use the second row to express y in terms of z as $y = -6 + 4z$. Now substitute into the first row $-x - (-6 + 4z) + 3z = 3$ to express x also in terms of z with $x = 3 - z$.

2.2 *Definition* In an echelon form linear system the variables that are not leading are *free*. A variable that we use to describe a family of solutions is a *parameter*.

We shall routinely parametrize linear systems using the free variables.

Example This system is already in echelon form.

$$\begin{aligned} 2x + \quad y + z - \quad w &= 5 \\ -y + z + 4w &= 6 \end{aligned}$$

The leading variables are x and y so we will parametrize the solution set with z and w. The second row gives $y = -6 + z + 4w$. Substituting into the first row gives $2x + (-6 + z + 4w) + z - w = 5$, so $x = (11/2) - z - (3/2)w$.

Example This is also already in echelon form.

$$\begin{aligned} -2x + y - \quad z + w &= 3/2 \\ 2z - w &= 1/2 \end{aligned}$$

We parametrize with y and w. The second row gives $z = 1/4 + (1/2)w$. Substituting back into the first row leaves $x = -(7/8) + (1/2)y + (1/4)w$.

$$\begin{aligned} x &= -(7/8) + (1/2)y + (1/4)w \\ y &= y \\ z &= 1/4 + (1/2)w \\ w &= w \end{aligned}$$