Math 124: Linear algebra
Friday October 14
9:40am - 10:30am, Perkins 107

① Return exams, go over any problem they are interested in seeing.

② End of Section Two.II (Linear independence)

Proposition
A set S is linearly independent if and only if for any $\vec{v} \in S$, its removal shrinks the span:

$$\underbrace{\text{Span } S \setminus \{\vec{v}\}}_{\text{remove } \vec{v}} \subsetneq \text{Span } S$$

Lemma
Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $\underbrace{S \cup \{\vec{v}\}}_{\text{add } \vec{v}}$ is linearly independent if and only if

$$\vec{v} \notin \text{Span } S$$

Moral:

Linear independence is exactly what decides if a vector is superfluous in a spanning set.

Example

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \right\}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

$\vec{v}_1 + \vec{v}_2 = \vec{v}_3$ so $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0$

i.e. linearly dependent

Then any one of these vectors is superfluous:

$$\begin{aligned} \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} &= \text{Span}\{\vec{v}_1, \vec{v}_2\} \\ &= \text{Span}\{\vec{v}_1, \vec{v}_3\} \\ &= \text{Span}\{\vec{v}_2, \vec{v}_3\} \end{aligned}$$

As far as what vectors we can make, we don't need all three.

Contrast:

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$\vec{v}_1 \qquad \vec{v}_2 \qquad \vec{v}_3$$

These 3 are linearly independent

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \overset{\rho_2 - \rho_1}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \overset{\rho_3 - \rho_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

all variables leading, unique solution

$$a_1 = a_2 = a_3 = 0.$$

All three vectors are necessary

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$$

$$\text{Span}\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \; y = x + z \right\}$$

$$\text{Span}\{\vec{v}_1, \vec{v}_3\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = y \right\}$$

$$\text{Span}\{\vec{v}_2, \vec{v}_3\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 0 \right\}$$

Corollary

In a vector space, any finite set has a linearly independent subset with the same span.

proof Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$

If $\text{Span}\, S = \text{Span}\, S \setminus \{\vec{v}_i\}$, remove that vector.

Repeat, remove vectors one by one until all are necessary. (i.e. removing one makes the span strictly smaller). What you are left with is linearly independent.

$\square$

How to do this in practice: ← get a lin indep subset

Say you have $S=\{\vec{v}_1, \dots, \vec{v}_n\}$ as in the proof.

1. Solve $r_1\vec{v}_1 + r_2\vec{v}_2 + \dots + r_n\vec{v}_n = \vec{0}$.
   If the only solution is $r_1 = r_2 = \dots = r_n = 0$, then the set is linearly independent already and we are done.

2. Otherwise we will get a solution set looking like

$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \vec{\beta}_1 r_1 + \vec{\beta}_3 r_3 + \vec{\beta}_4 r_4 \right\}$$

maybe i.e. some of the r_i's will be free.

Then for each i such that r_i is free, we can throw out $\vec{v}_i$. The leftover set is linearly independent.

In other words, the vectors $\vec{v}_i$ corresponding to leading r_i's are necessary only.

Example

Let $S = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\qquad\qquad \overset{\shortparallel}{\vec{v}_1} \qquad \overset{\shortparallel}{\vec{v}_2} \qquad \overset{\shortparallel}{\vec{v}_3} \qquad \overset{\shortparallel}{\vec{v}_4} \qquad \overset{\shortparallel}{\vec{v}_5}$

Give a subset of S that is linearly independent but that has the same span.

Solution: We solve

$$r_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} + r_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + r_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + r_5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 1 & 0 & 1 \\ 2 & 3 & 4 & -1 & -1 \end{pmatrix} \overset{\rho_2 - \rho_1}{\sim} \begin{pmatrix} 2 & 3 & 1 & 0 & 1 \\ 0 & 0 & 3 & -1 & -2 \end{pmatrix}$$

$$\overset{\frac{1}{3}\rho_2}{\sim} \begin{pmatrix} 2 & 3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/3 & -2/3 \end{pmatrix}$$

$$\overset{\rho_1 - \rho_2}{\sim} \begin{pmatrix} 2 & 3 & 0 & 1/3 & 5/3 \\ 0 & 0 & 1 & -1/3 & -2/3 \end{pmatrix} \overset{\frac{1}{2}\rho_1}{\sim} \begin{pmatrix} 1 & 3/2 & 0 & 1/6 & 5/6 \\ 0 & 0 & 1 & -1/3 & -2/3 \end{pmatrix}$$

r_1 and r_3 are leading so

$$\text{Span}\{\vec{v}_1, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$$

and $\{\vec{v}_1, \vec{v}_3\}$ is linearly independent.

Why ??

Well
$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \right\}$$

If $r_2 = 2$, $r_4 = 0$, $r_5 = 0$,

then $r_1 = -3$, $r_3 = 0$

so $-3\vec{v}_1 + 2\vec{v}_2 = 0$ & $\vec{v}_2$ is superfluous.

If $r_2 = 0$, $r_4 = 6$, $r_5 = 0$

then $r_1 = -1$, $r_3 = 2$

so $-\vec{v}_1 + 2\vec{v}_3 + 6\vec{v}_4 = 0$ & $\vec{v}_4$ is superfluous

If $r_2 = 0$, $r_4 = 0$, $r_5 = 6$

then $r_1 = -5$, $r_3 = 4$

so $-5\vec{v}_1 + 4\vec{v}_3 + 6\vec{v}_5 = 0$ & $\vec{v}_5$ is superfluous i.e. we can "eliminate" $\vec{v_2}$, $\vec{v_4}$ and $\vec{v_5}$ one by one by taking

$$r_2 \neq 0 \quad r_4 = 0 \quad r_5 = 0$$
$$r_2 = 0 \quad r_4 \neq 0 \quad r_5 = 0$$
$$r_2 = 0 \quad r_4 = 0 \quad r_5 \neq 0$$