Three.III Computing Linear Maps

Linear Algebra

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Representing Linear Maps with Matrices

Linear maps are determined by the action on a basis

Fix a domain space V with basis $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$, and a codomain space W. We've seen that to specify the action of a homomorphism $h\colon V\to W$ on all domain vectors, we need only specify its action on the basis elements.

$$h(\vec{v})=h(c_1\cdot\vec{\beta}_1+\cdots+c_n\cdot\vec{\beta}_n)=c_1\cdot h(\vec{\beta}_1)+\cdots+c_n\cdot h(\vec{\beta}_n)\quad(*)$$

We've called this extending the action linearly from the basis to the entire domain. We now introduce a scheme for these calculations.

Example Let the domain be $V=\mathcal{P}_2$ and the codomain be $W=\mathbb{R}^2$, with these bases.

$$B_V=\langle 1,1+x,1+x+x^2\rangle\qquad B_W=\langle\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\rangle$$

Suppose that $h\colon\mathcal{P}_2\to\mathbb{R}^2$ has this action on the domain basis.

$$h(1)=\begin{pmatrix}0\\1\end{pmatrix}\quad h(1+x)=\begin{pmatrix}3\\2\end{pmatrix}\quad h(1+x+x^2)=\begin{pmatrix}-2\\-1\end{pmatrix}$$

Example Again consider projection onto the x-axis

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

but this time take the input and output bases to be the standard.

$$B = D = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this is $\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi)$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example Consider the domain $\mathbb{R}^2$ and the codomain $\mathbb{R}$. Recall that with respect to the standard basis, a vector represents itself.

$$\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix} -2 \\ 2 \end{pmatrix}) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}_{\mathcal{E}_2}$$

To represent $h\colon \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{h}{\longmapsto} 2a + 3b$$

with respect to $\mathcal{E}_2$ and $\mathcal{E}_1$, first find the effect of h on the domain's basis.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto 2 \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto 3$$

Represent those with respect to the codomain's basis.

$$\mathrm{Rep}_{\mathcal{E}_1}(h(\vec{e}_1)) = \begin{pmatrix} 2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_1}(h(\vec{e}_2)) = \begin{pmatrix} 3 \end{pmatrix}$$

This is 1×2 matrix representation.

$$H = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_1}(h) = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

Proof This formalizes the example that began this subsection. See Exercise 32 . QED

1.5 *Definition* The *matrix-vector product* of a $m \times n$ matrix and a $n \times 1$ vector is this.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{1,1}c_1 + \cdots + a_{1,n}c_n \\ a_{2,1}c_1 + \cdots + a_{2,n}c_n \\ \vdots \\ a_{m,1}c_1 + \cdots + a_{m,n}c_n \end{pmatrix}$$

Example We can perform the operation without any reference to spaces and bases.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 + 1 \cdot (-1) + 2 \cdot (-3) \\ 0 \cdot 4 - 2 \cdot (-1) + 5 \cdot (-3) \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

Example Recall also that the map $h\colon \mathbb{R}^2 \to \mathbb{R}$ with this action

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{h}{\longmapsto} 2a + 3b$$

is represented with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_1$ by a 1×2 matrix.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_1}(h) = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

The domain vector

$$\vec{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

has this image.

$$\mathrm{Rep}_{\mathcal{E}_1}(h(\vec{v})) = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}_{\mathcal{E}_1}$$

Since this is a representation with respect to the standard basis $\mathcal{E}_1$, meaning that vectors represent themselves, the image is $h(\vec{v}) = 2$.