MATH 124: Homework 4 Solutions

#1 This question is about the space of all matrices that look like

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

a) the dimension is 6

b) $$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The set

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

is a spanning set, as shown above. It is also linearly independent. This is because at each position of the matrix, only one matrix of the spanning set has a nonzero entry.

#2a) Let $\vec{v}_1 = \begin{pmatrix} a_1+b_1 \\ a_1+c_1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} a_2+b_2 \\ a_2+c_2 \end{pmatrix} \in V$

Then $$r_1\vec{v}_1 + r_2\vec{v}_2 = \begin{pmatrix} r_1(a_1+b_1) + r_2(a_2+b_2) \\ r_1(a_1+c_1) + r_2(a_2+c_2) \end{pmatrix}$$

$$= \begin{pmatrix} (r_1a_1+r_2a_2) + (r_1b_1+r_2b_2) \\ (r_1a_1+r_2a_2) + (r_1c_1+r_2c_2) \end{pmatrix} \in V$$

with $a = r_1a_1 + r_2a_2$ $\quad b = r_1b_1 + r_2b_2$ $\quad c = r_1c_1 + r_2c_2$

b) The dimension is 2 (see c) for justification)

c) $$\begin{pmatrix} a+b \\ a+c \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for V

We shrink to a basis:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \overset{\rho_2 - \rho_1}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

this is echelon form the first 2 columns are leading, the last is free.

Therefore the last vector of the spanning set is superfluous and a basis is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

Justification: Our method is guaranteed to give a basis