Math 124 - Homework 2 solutions

#1 a) $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$

#2 To give a plane, we need one point on the plane and 2 directions of motion.

1st direction: $\begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix}$

2nd direction: $\begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 5 \\ -5 \end{pmatrix}$

point on the plane: $\begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix}$

Equation: $\begin{pmatrix} 1 \\ 1 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix} s + \begin{pmatrix} -2 \\ 0 \\ 5 \\ -5 \end{pmatrix} t, \quad s, t \in \mathbb{R}$

#3a) We know that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$$

if θ is the angle between $\vec{u}$ & $\vec{v}$.

We have that $\cos\frac{\pi}{2} = \cos -\frac{\pi}{2} = 0$, so

if 2 vectors are perpendicular,

then $$0 = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}.$$

A fraction is 0 if its numerator is 0,

so if $\vec{u}$ x $\vec{v}$ are perpendicular, then

$$\vec{u}\cdot\vec{v} = 0.$$

b) i. S is a subset of $\mathbb{R}^3$, which is a vector space. So we only need to show that it is a subspace.

To show that S is a subspace, it suffices to show that if $\vec{a}$ and $\vec{b}$ are elements of S, then $r\vec{a}+s\vec{b}$ is also in S for any $r, s \in \mathbb{R}$.

In other words, we must show that if $\vec{a}$ and $\vec{b}$ are both perpendicular to $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, then so is $r\vec{a}+s\vec{b}$.

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. They are both perpendicular to $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ so

$$\boxed{\begin{aligned} \vec{a}\cdot\vec{v} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = a_1 + 2a_2 = 0 \\ \vec{b}\cdot\vec{v} &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = b_1 + 2b_2 = 0. \end{aligned}}$$

Now $r\vec{a}+s\vec{b} = \begin{pmatrix} ra_1+sb_1 \\ ra_2+sb_2 \\ ra_3+sb_3 \end{pmatrix}$.

Let's compute

$$(r\vec{a}+s\vec{b})\cdot\vec{v} = \begin{pmatrix} ra_1+sb_1 \\ ra_2+sb_2 \\ ra_3+sb_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= (ra_1+sb_1)+2(ra_2+sb_2)$$

$$= ra_1+2ra_2+sb_1+2sb_2$$

$$= r(a_1+2a_2)+s(b_1+2b_2)$$

$$= r\cdot 0 + s\cdot 0 = 0$$

S is indeed a vector space

ii. $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in S$ if $\vec{u}\cdot\vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = u_1+2u_2=0$

We solve $u_1+2u_2=0$ for u_1, u_2 and u_3.
We see that only u_1 is a leading variable (there is only one equation) and u_2 and u_3 are free.

We get $u_1 = -2u_2$

$u_2 = u_2$

$u_3 = u_3$

In vector form this is $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_3$

Therefore S is spanned by $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$S = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

#4 Consider a system of m equations in n variables, and assume that this system is homogeneous.

Write $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = 0$

for its ith equation.

Since these equations have n variables, the set of solutions is contained in $\mathbb{R}^n$, which is a vector space, again we only need to check that the solutions form a subspace.

To do this, let $\vec{s} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$ and $\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$

be solutions and r_1 and r_2 be two real numbers. We must show $r_1\vec{s} + r_2\vec{t}$ is also a solution. Notice that

$$r_1\vec{s} + r_2\vec{t} = \begin{pmatrix} r_1 s_1 + r_2 t_1 \\ r_1 s_2 + r_2 t_2 \\ \vdots \\ r_1 s_n + r_2 t_n \end{pmatrix}$$

To show it is a solution, we plug in $r_1\vec{s} + r_2\vec{t}$ into each equation of the system, i.e. in the ith equation for $i = 1, 2, 3, \dots, m$

This looks like this:

$$a_{i1}(r_1s_1+r_2t_1)+a_{i2}(r_1s_2+r_2t_2)+\dots$$
$$+a_{in}(r_1s_n+r_2t_n)$$

$$=a_{i1}r_1s_1+a_{i2}r_1s_2+\dots+a_{in}r_1s_n$$
$$+a_{i1}r_2t_1+a_{i2}r_2t_2+\dots+a_{in}r_2t_n$$

$$=r_1(a_{i1}s_1+a_{i2}s_2+\dots+a_{in}s_n)$$
$$+r_2(a_{i1}t_1+a_{i2}t_2+\dots+a_{in}t_n)$$

$$=r_1 0+r_2 0=0$$

and the quantities in parentheses are 0 because both $\vec{s}$ and $\vec{t}$ are assumed to be solutions. Since this works for $i=1,2\dots m$, $r_1\vec{s}+r_2\vec{t}$ is a solution of the whole system, and the set of solutions of a homogeneous system of linear equations is a vector space.

If the system is not homogeneous though, then the set of solutions is <u>not</u> a vector space.

This time, we <u>choose</u> i such that the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = d_i \neq 0$$

We know there is at least one like this, otherwise the system would be homogeneous.

Then the set of solutions is not a vector space because if $\vec{s}$ is a solution, then $2\vec{s}$ is <u>not</u> a solution:

If $a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = d_i$, then

$$a_{i1}2s_1 + a_{i2}2s_2 + \dots + a_{in}2s_n$$
$$= 2(a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n)$$
$$= 2d_i \neq d_i$$

(Also $\vec{0}$ is not in the set of solutions, etc. This set is not a vector space in many ways!)