Math 124: Fall 2016
Final Exam

NAME: SOLUTIONS

Time: **2 hours 45 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score
1	10	
2	4	
3	8	
4	5	
5	5	
6	8	
7	6	
8	9	
9	5	
10	5	
11	10	
12	10	
13	15	
TOTAL	100	

Problem 1 : (10 points) Solve each of the following systems of linear equations. If you do find solution(s), check your answer.

a)

$$\begin{aligned} x \qquad - z &= 1 \\ 2x + y \qquad &= 1 \\ 2x + 2y + 2z &= 1 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 \end{array}\right) \underset{\rho_3 - 2\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -1 \end{array}\right)$$

$$\overset{\rho_3 - 2\rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right) \leftarrow \text{this is } 0 = 1, \text{ a contradiction}$$

There are no solutions to this system.

b)

$$\begin{aligned} x \quad\quad - z \quad\quad &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 1 & 2 & 3 & -1 & 7 \end{array}\right) \overset{\rho_3-\rho_1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 2 & 4 & -1 & 6 \end{array}\right)$$

$$\overset{\rho_3-2\rho_2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\overset{\rho_2+\rho_3}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

We go to reduced echelon form because we are solving

z is free

$x - z = 1$

$y + 2z = 3$

$w = 0$

so $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

check particular solution:

$1 - 0 = 1$ ✓

$3 + 0 = 3$ ✓

$1 + 2 \cdot 3 = 7$ ✓

check homogeneous solution

$z - z = 0$ ✓

$-2z + 2z = 0$ ✓

$z + 2(-2z) + 3z = 0$ ✓

Problem 2 : (4 points) Consider the set

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

Is this set linearly dependent or linearly independent?

We solve $a_1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} a_1 = 0 \\ a_2 = 0 \end{matrix}$$

This is the unique solution so the vectors are linearly independent.

Problem 3 : (8 points) Consider the set

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}.$$

a) Prove that B is a basis for $\mathbb{R}^2$.

Show B is linearly independent: $a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ unique solution $a_1 = a_2 = 0$

B is linearly independent so it spans a space of dimension 2 inside $\mathbb{R}^2$. The only such space is all of $\mathbb{R}^2$ so B spans $\mathbb{R}^2$ and B is a basis for $\mathbb{R}^2$.

b) For $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, give $\text{Rep}_B(\vec{v})$.

Find a_1, a_2 with $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right)$

$a_1 = 5$ $a_2 = -3$ so

$\text{Rep}_B(\vec{v}) = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

check

$5\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10-9 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ✓

Problem 4 : (5 points) Consider the homogeneous system of linear equations

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

What is the dimension of its solution set? Support your answer by giving a basis. Be sure to argue that you have found a basis.

Solve the system and give answer in vector form

$$\begin{pmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_3, x_4 are free
x_1 is leading

$$x_1 - 4x_2 + 3x_3 - x_4 = 0 \rightsquigarrow x_1 = 4x_2 - 3x_3 + x_4$$

Solutions
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4$$

The dimension of the solution set is 3

A basis for the solution set is $\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

This is linearly independent; it can be seen by looking at the last three coordinates.

Problem 5 : (5 points) Is the vector

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

in the column space of the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}?$$

Is there a solution to $a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_3\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$?

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 4 & 3 \end{array}\right) \overset{\rho_1-\rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 3 \end{array}\right)$$

$a_1 - a_3 = -2$ $\quad a_1 = a_3 - 2$

$a_2 + 4a_3 = 3$ $\quad a_2 = -4a_3 + 3$

a_3 is free

solutions: $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} a_3 + \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$

Yes, there is a solution (in fact there are infinitely many!)

so $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is in the column space of this matrix

check $\quad a_1 = -2$, $a_2 = 3$, $a_3 = 0$ $\qquad -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ✓

Problem 6 : (8 points) Consider the matrix

$$\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}.$$

a) Give a basis for the row space of this matrix.

Shrink the set $\left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix} \right\}$ to a basis

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 2 \\ 3 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 3 & 1 & 7 \end{pmatrix} \overset{\rho_3 - 3\rho_1}{\sim} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \underset{\rho_3 + \rho_2}{\overset{-\rho_2}{\sim}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \ \uparrow$ leading

A basis for the row space is

$\left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ our method guarantees that this is a basis.

b) Is the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the row space of this matrix?

It is enough to check if there is a solution to

$$a_1 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Augmented matrix: $\left(\begin{array}{cc|c} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 3 & 1 & 1 \end{array}\right) \overset{\rho_3 - 3\rho_1}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \end{array}\right)$

$$\underset{\rho_3 + \rho_2}{\overset{-\rho_2}{\sim}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{array}\right)$$

this is a contradiction $0 = -1$

$\leftarrow$ so there is no solution

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is not in the row space

Note: Throwing in $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ would not help since it is dependent and therefore cannot make anything the other two can't.

Problem 7 : (6 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the homomorphism such that

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}, \quad \text{and} \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

a) What is the matrix representation of f?

$$\begin{pmatrix}2 & 0\\2 & 1\\0 & -1\end{pmatrix}$$

b) Give the formula for $f\begin{pmatrix}x\\y\end{pmatrix}$.

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & 0\\2 & 1\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\2x+y\\-y\end{pmatrix}$$

c) What is $f\begin{pmatrix}-2\\1\end{pmatrix}$?

$$f\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}2(-2)\\2(-2)+1\\-1\end{pmatrix} = \begin{pmatrix}-4\\-3\\-1\end{pmatrix}$$

Problem 8 : (9 points) Perform the following matrix operations if they are defined. If they are not defined, state "not defined."

a) $4\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix} + 5\begin{pmatrix}-1 & 1\\ -2 & 1\end{pmatrix}$

$$= \begin{pmatrix}4 & 8\\ 12 & -4\end{pmatrix} + \begin{pmatrix}-5 & 5\\ -10 & 5\end{pmatrix} = \begin{pmatrix}-1 & 13\\ 2 & 1\end{pmatrix}$$

b) $\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & -1\\ 4 & 0 & 3\end{pmatrix}$

$$= \begin{pmatrix}2\cdot1-1\cdot4 & 2\cdot1 & 2\cdot(-1)-1\cdot3\\ 3\cdot1+1\cdot4 & 3\cdot1 & 3\cdot(-1)+1\cdot3\end{pmatrix} = \begin{pmatrix}-2 & 2 & -5\\ 7 & 3 & 0\end{pmatrix}$$

c) $\begin{pmatrix}1 & 1 & -1\\ 4 & 0 & 3\end{pmatrix}\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}$

$2\times\underbrace{3 \cdot 2}_{\text{not equal}}\times 2$

This multiplication is not defined.

Problem 9 : (5 points) Compute the inverse of this matrix, if it exists.

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ 4 & -2 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 \\ 4 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \rho_2 - 2\rho_1 \\ \sim \\ \rho_3 - 4\rho_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 6 & 9 & 1 & -2 & 0 \\ 0 & 6 & 9 & 0 & -4 & 1 \end{array}\right)$$

$$\overset{\rho_3 - \rho_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 6 & 9 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right)$$

we can't ever get to the identity matrix. the inverse does not exist.

Problem 10 : (5 points) Compute the determinant of the following matrix:

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

expand along this column

$$(-1)^{2+1} \cdot 0 + (-1)^{2+2} \cdot 0 + (-1)^{2+3} \cdot 3 \begin{vmatrix} 4 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{vmatrix} + (-1)^{2+4} \cdot 0$$

expand along this Row

$$= -3 \left((-1)^{1+2} \cdot 0 + (-1)^{2+2} \cdot 1 \cdot \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} + (-1)^{3+2} \cdot 0 \right)$$

$$= -3 \cdot 1 \cdot 1 \ (4 \cdot 2 - 2 \cdot 2)$$

$$= -3(8-4) = -3 \cdot 4 = -12$$

Problem 11 : (10 points) Consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Find all of the eigenvalues of this matrix.

They are the roots of

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(-\lambda)(1-\lambda)$$

expand along this column

The roots are $\lambda_1 = 1$ and $\lambda_2 = 0$

b) For each eigenvalue, find a basis for the eigenspace.

$\underline{\underline{\lambda_1=1}}$ Solve $\begin{pmatrix} 1-1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

x is free, y and z are leading $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x$ Basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\underline{\underline{\lambda_2=0}}$ Solve $\begin{pmatrix} 1-0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{matrix} x+y=0 \\ z=0 \end{matrix}$

y is free, x and z are leading $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} y$ Basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Both bases are linearly independent because a single nonzero vector is linearly independent.

Problem 12 : (10 points) Let

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

$\vec{u} \cdot \vec{v} = 0$

a) Prove that the set of all vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$ is a vector space.

Since this is inside $\mathbb{R}^3$, which is a vector space, it is enough to show that if both $\vec{u}_1 \cdot \vec{v} = 0$ and $\vec{u}_2 \cdot \vec{v} = 0$ and r_1 and $r_2 \in \mathbb{R}$, then $(r_1\vec{u}_1 + r_2\vec{u}_2) \cdot \vec{v} = 0$

Let $\vec{u}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\vec{u}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$\vec{u}_1 \cdot \vec{v} = x_1 + 2y_1 = 0$

$\vec{u}_2 \cdot \vec{v} = x_2 + 2y_2 = 0$

$$(r_1\vec{u}_1 + r_2\vec{u}_2) \cdot \vec{v} = \begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= (r_1x_1 + r_2x_2) + 2(r_1y_1 + r_2y_2)$$

$$= r_1(x_1 + 2y_1) + r_2(x_2 + 2y_2) = 0,$$

b) Give a basis for this vector space.

This vector space is $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x + 2y = 0$

z is free

so $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z$$

A basis is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

This can be seen to be linearly independent by looking at the last 2 coordinates.

Problem 13 : (15 points) Consider the homomorphism

$$f \colon \mathbb{R}^3 \mapsto \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - z \\ 2x + y \\ 2x + 2y + 2z \end{pmatrix}.$$

a) (2 points) Give the matrix representation of f. Call this matrix A.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

b) (3 points) Give the reduced echelon form of the matrix A.

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix} \begin{matrix} \rho_2 - 2\rho_1 \\ \sim \\ \rho_3 - 2\rho_1 \end{matrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \overset{\rho_3 - 2\rho_2}{\sim} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- all leading ones
- zeroes above and below leading ones.

For your convenience, here is the formula for f again:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 2x + y \\ 2x + 2y + 2z \end{pmatrix}.$$

c) (4 points) Compute the nullity of f. Support your answer by giving a basis for the null space of f.

We want x, y, z with $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We already know the reduced echelon form of A

So this is $x - z = 0 \rightsquigarrow x = z$

$y + 2z = 0 \rightsquigarrow y = -2z$

z is free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} z$

The nullity is 1. A basis for the null space is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

This is a basis because a single nonzero vector is linearly independent.

For your convenience, here is the formula for f again:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 2x + y \\ 2x + 2y + 2z \end{pmatrix}.$$

d) (4 points) Compute the rank of f. Support your answer by giving a basis for the range space of f.

The range space of f is the column space of A.

We know that in the echelon form of A, the first 2 columns are leading, so a basis for the column space / range space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

The method guarantees this is a basis.

The rank of f is 2.

e) (2 points) Is f an isomorphism? Support your answer using your work in parts c) and d) of this question.

No. f is not one-to-one because its nullity is not 0 and it is not onto because its rank is not 3.