Math 124: Fall 2016
Final Exam

NAME:

Time: **2 hours 45 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score
1	10	
2	4	
3	8	
4	5	
5	5	
6	8	
7	6	
8	9	
9	5	
10	5	
11	10	
12	10	
13	15	
TOTAL	100	

Problem 1 : (10 points) Solve each of the following systems of linear equations. If you do find solution(s), check your answer.

a)

$$\begin{aligned} x \quad\quad - \; z &= 1 \\ 2x + \; y \quad\quad &= 1 \\ 2x + 2y + 2z &= 1 \end{aligned}$$

b)

$$\begin{aligned} x \qquad - \;\; z \qquad &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

Problem 2 : (4 points) Consider the set

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

Is S linearly dependent or linearly independent?

Problem 3 : (8 points) Consider the set

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}.$$

a) Prove that B is a basis for $\mathbb{R}^2$.

b) For $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, give $\mathrm{Rep}_B(\vec{v})$.

Problem 4 : (5 points) Consider the homogeneous system of linear equations

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

What is the dimension of its solution set? Support your answer by giving a basis. Be sure to argue that you have found a basis.

Problem 5 : (5 points) Is the vector

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

in the column space of the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}?$$

Problem 6 : (8 points) Consider the matrix

$$\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}.$$

a) Give a basis for the row space of this matrix. Be sure to argue that you have found a basis.

b) Is the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the row space of this matrix?

Problem 7 : (6 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the homomorphism such that

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}, \quad \text{and} \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

a) What is the matrix representation of f?

b) Give the formula for $f\begin{pmatrix}x\\y\end{pmatrix}$.

c) What is $f\begin{pmatrix}-2\\1\end{pmatrix}$?

Problem 8 : (9 points) Perform the following matrix operations if they are defined. If they are not defined, state "not defined."

a) $4\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} + 5\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$

Problem 9 : (5 points) Compute the inverse of this matrix, if it exists.

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$$

Problem 10 : (5 points) Compute the determinant of the following matrix:

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Problem 11 : (10 points) Consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Find all of the eigenvalues of this matrix.

b) For each eigenvalue, find a basis for the eigenspace. Be sure to argue that you have found a basis.

Problem 12 : (10 points) Let

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

a) Prove that the set of all vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$ is a vector space.

b) Give a basis for this vector space.

Problem 13 : (15 points) Consider the homomorphism

$$f\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - z \\ 2x + y \\ 2x + 2y + 2z \end{pmatrix}.$$

a) (2 points) Give the matrix representation of f. Call this matrix A.

b) (3 points) Give the reduced echelon form of the matrix A.

For your convenience, here is the formula for f again:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 2x + y \\ 2x + 2y + 2z \end{pmatrix}.$$

c) (4 points) Compute the nullity of f. Support your answer by giving a basis for the null space of f. Be sure to argue that you have found a basis.

For your convenience, here is the formula for f again:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 2x + y \\ 2x + 2y + 2z \end{pmatrix}.$$

d) (4 points) Compute the rank of f. Support your answer by giving a basis for the range space of f. Be sure to argue that you have found a basis.

e) (2 points) Is f an isomorphism? Support your answer using your work in parts c) and d) of this question.