Change of basis

Remember the matrix $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

Eigenvalue $\lambda_1 = 1$ basis for eigenspace $\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

$\lambda_2 = -3$ basis for eigenspace $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

<u>Fact</u> Putting together all of the bases for all of the eigenspaces gives a set that is still linearly independent:

$$B = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is 3 vectors that are linearly independent in $\mathbb{R}^3$

$\Rightarrow$ B is a basis for $\mathbb{R}^3$

If we can find a basis for everything made only of eigenvectors (i.e. if we have enough eigenvectors) we say A is diagonalizable. Why?

Since B is a basis for $\mathbb{R}^3$, for any $\vec{v}$ in $\mathbb{R}^3$ there are a_1, a_2, a_3 with

$$\vec{v} = a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

We write $\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Now remember that $A \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ eigenvector with eigenvalue 1

$A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$A \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

eigenvectors with eigenvalue -3

So
$$A\vec{v} = A\left(a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= a_1 A \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 A \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$= a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} - 3a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 3a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

In other words

$$\mathrm{Rep}_B(A\vec{v}) = \begin{pmatrix} a_1 \\ -3a_2 \\ -3a_3 \end{pmatrix}$$

In other other words, if we only worked with the basis B instead of the standard basis

A would be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ ← a diagonal matrix

A is diagonalizable because there is a basis on which it acts like a diagonal matrix!

$$\begin{tikzcd}
\vec{v} \arrow[d, mapsto] & \underset{\text{standard basis}}{\mathbb{R}^3} \arrow[r, "{\begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}}"] \arrow[d] & \underset{\text{standard basis}}{\mathbb{R}^3} & b_1\begin{pmatrix}-2\\-1\\1\end{pmatrix} + b_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + b_3\begin{pmatrix}-2\\0\\1\end{pmatrix} \\
\mathrm{Rep}_B(\vec{v}) & \underset{\text{basis } B}{\mathbb{R}^3} \arrow[r, "{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}}"] & \underset{\text{basis } B}{\mathbb{R}^3} & \text{if } \mathrm{Rep}_B(\vec{w}) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \arrow[u, mapsto]
\end{tikzcd}$$

Let's look at the vertical arrow on the left

If $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in the standard basis

then $\mathrm{Rep}_B \vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ where a_1, a_2, a_3 are the

solutions to $a_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e. if $C = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ we solve $C\vec{a} = \vec{x}$ for $\vec{a}$

i.e. $\vec{a} = C^{-1}\vec{x}$

Key insight: The map $\vec{v} \mapsto \mathrm{Rep}_B \vec{v}$ is given by the matrix C^{-1}

C^{-1} is a change-of-basis matrix. It changes from the standard basis to the basis B.

Now let's look at the vertical arrow on the Right

It does
$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mapsto b_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -2b_1 - b_2 - 2b_3 \\ -b_1 + b_2 \\ b_1 + b_3 \end{pmatrix}$$
$$= C\vec{b}$$

Key insight: Going back from the basis $\mathcal{B}$ to the standard basis is given by the matrix $C = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

It makes sense: C and C^{-1} undo each other

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, bend left, "C^{-1}"] & \mathbb{R}^3 \arrow[l, bend left, "C"] \end{tikzcd}$$

standard basis   basis $\mathcal{B}$

<u>Every</u> invertible matrix is a change-of-basis matrix

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ goes from $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ to standard basis

A^{-1} goes from standard basis to basis $\mathcal{B}$

Write $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & 11 \end{pmatrix}$ $C = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

our diagram was

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "A"] \arrow[d, "C^{-1}"'] & \mathbb{R}^3 \\ \mathbb{R}^3 \arrow[r, "D"] & \mathbb{R}^3 \arrow[u, "C"'] \end{tikzcd}$$

So $A = CDC^{-1}$ or $D = C^{-1}AC$

Since A & D are related by a change-of-basis, we say A and D are similar matrices

Remarks

- If A is not diagonalizable, A is not similar to any diagonal matrix, but it does have a Jordan Canonical Form (diagonal with maybe some 1's above the diagonal) That's almost as nice.

- Fun thing we can do:

$$A^{10} = (CDC^{-1})^{10} = CD\underbrace{C^{-1}C}_{\text{cancel}}D\underbrace{C^{-1}C}_{\text{cancel}}DC^{-1} \;\ldots\; CDC^{-1}$$
$$\ldots$$
$$= CD^{10}C^{-1}$$

but $D^{10} = \begin{pmatrix} 1^{10} & 0 & 0 \\ 0 & (-3)^{10} & 0 \\ 0 & 0 & (-3)^{10} \end{pmatrix}$ ← easy to compute!

- What if you want to go between

$$\text{Rep}_{B_1}(\vec{v}) \quad \text{and} \quad \text{Rep}_{B_2}(\vec{v})$$

where B_1 and B_2 are 2 bases but neither is the standard basis?

Let C_1 go between B_1 & standard basis

C_2 go between B_2 & standard basis

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, shift right, "C_1"'] & \mathbb{R}^n \arrow[l, shift right, "C_1^{-1}"'] \arrow[r, shift right, "C_2^{-1}"'] & \mathbb{R}^n \arrow[l, shift right, "C_2"']
\end{tikzcd}$$

with basis B_1 | with standard basis | with basis B_2

Then $C_2^{-1}C_1$ is the matrix that converts from B_1 to B_2

and $C_1^{-1}C_2$ is the matrix that converts from B_2 to B_1