Subgraph complementation and minimum rank

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Subgraph complementation

A *subgraph complementation* of a simple graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ w.r.t. $\mathsf{U} \subseteq \mathsf{V}$ is the operation of complementing the edges and non-edges of the induced subgraph $\mathsf{G}[\mathsf{U}]$.

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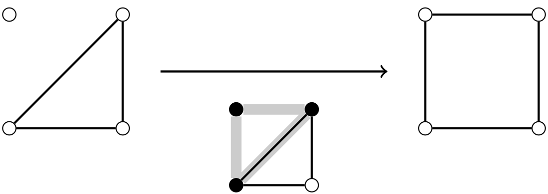

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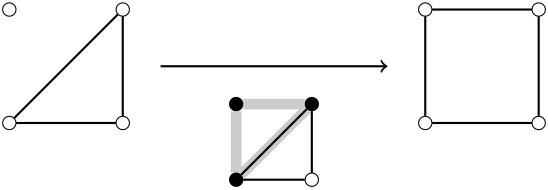

Observations

- Any graph H on V can be obtained from G via a sequence of subgraph complementations.
- The same sequence of complementations applied to the graph with no edges builds the symmetric difference $\mathsf{G} \triangle \mathsf{H}$.

Subgraph complementation systems

Question

What is the minimum number of subgraph complementations required to obtain G *from the graph on* V *with no edges?*

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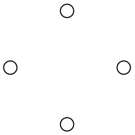

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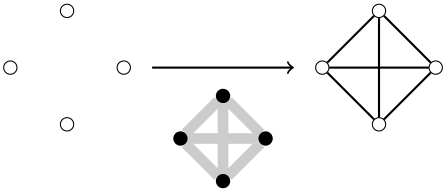

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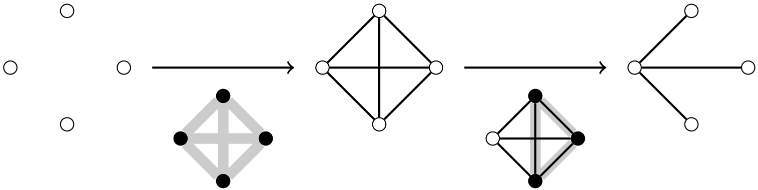

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Definitions

Subgraph complementation system of G:
A collection $\mathscr{C}$ of subsets of V with the property that G is obtained from the empty graph by successive subgraph complementations w.r.t. the sets in $\mathscr{C}$.

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- adjacent vertices in G are in an odd number of sets together
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Subgraph complementation number of G, $\mathsf{c}_2(\mathsf{G})$:
The minimum cardinality of a subgraph complementation system of G.

Equivalent problems

1. Minimum number of complete graphs $G_1, \dots, G_k$ with the property that any edge of G is in an odd number of G_i's, and any non-edge in an even number.[1]

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Equivalent problems

1. Minimum number of complete graphs $G_1, \ldots, G_k$ with the property that any edge of G is in an odd number of G_i's, and any non-edge in an even number.[1]

2. Faithful orthogonal representations over $\mathbb{F}_2$.[2]

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[2] L. Lovász. On the Shannon capacity of a graph. IEEE Transactions on Information Theory. 25(1):1–7 (1979)

Faithful orthogonal representations

Faithful orthogonal representation of G *over* $\mathbb{F}$ *of dimension* d:

A map $f : V(G) \to \mathbb{F}^d$ such that

$$uv \notin E(G) \iff f(u) \perp f(v).$$

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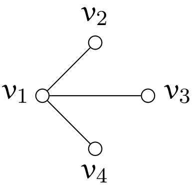
Example ($\mathbb{F} = \mathbb{F}_2$)

$\mathsf{G} =$ 

$$\mathsf{f}(v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathsf{f}(v_2) = \mathsf{f}(v_3) = \mathsf{f}(v_4) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Incidence matrices and orthogonal representations

Let $\mathscr{C} = C_1, \ldots, C_d$ be a subgraph complementation system for a graph G with vertices $v_1, \ldots, v_n$.

Define $M(\mathscr{C}) = (m_{i,j})$ to be the $n \times d$ matrix over $\mathbb{F}_2$ with entry

$$m_{i,j} = \begin{cases} 1 & \text{if } v_i \in C_j; \\ 0 & \text{otherwise.} \end{cases}$$

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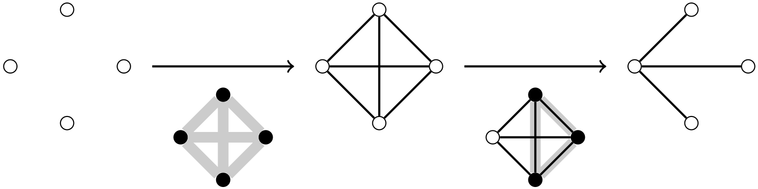
Example (a subgraph complementation system $\mathscr{C}$ for $K_{1,3}$)



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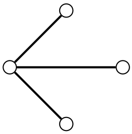
Example (of an incidence matrix for $\mathscr{C}$)

$\mathsf{G} =$ 

$$\mathsf{M}(\mathscr{C}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Some upper bounds

A number of upper bounds are known for the minimum dimension of a faithful orthogonal representation,[1] which we obtain as corollaries for $c_2(G)$:

- $c_2(G) \leqslant n - 1$ for all n-vertex G.
- $c_2(G) \leqslant n - 2$ if G is not a path on n vertices.

[1] V. Alekseev and V. Lozin, On orthogonal representations of graphs, *Discrete Mathematics*, 2001.

Minimum rank

Let G be a graph, $\mathscr{C}$ a subgraph complementation system of G, and $M = M(\mathscr{C})$.

The matrix $A = (a_{ij})$ given by $A = MM^T \pmod 2$ has the property that

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A matrix with this property is said to *fit* G.

The *minimum rank* of G over $\mathbb{F}$, $\mathrm{mr}(\mathsf{G}, \mathbb{F})$, is the minimum rank over all symmetric matrices over $\mathbb{F}$ which fit G.

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Proposition (BPR)

For any graph G*, we have* $\mathrm{mr}(\mathsf{G}, \mathbb{F}_2) \leqslant c_2(\mathsf{G})$.

Matrix factorization

Lemma (Friedland, Loewy, 2012)

For any matrix $\mathsf{A} \in \mathbb{F}_2^{\mathsf{n}\times\mathsf{n}}$ *of rank* k*, there exists a matrix* $\mathsf{X} \in \mathbb{F}_2^{\mathsf{n}\times\mathsf{k}}$ *such that either*

1. $A = \mathsf{X}\mathsf{X}^\mathsf{T}$, *or*
2. $A = \mathsf{X}\left(\oplus_1^{\mathsf{l}} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)\mathsf{X}^\mathsf{T}$, *where* $\mathsf{k} = 2\mathsf{l}$.

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Corollary

If $\mathrm{mr}(G, \mathbb{F}_2)$ *is odd, then* $c_2(G) = \mathrm{mr}(G, \mathbb{F}_2)$*.*

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We are able to strengthen these statements:

With A and X as above, we have

1. $A = XX^T \iff a_{i,i} = 1$ for some i, or
2. $A = X\left(\oplus_1^l \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\right)X^T \iff a_{i,i} = 0$ for all i.

Minimum rank

Theorem (BPR)

For any graph G,

$$\mathrm{mr}(\mathsf{G}, \mathbb{F}_2) \leqslant \mathsf{c}_2(\mathsf{G}) \leqslant \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1.$$

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Proof.

If $\mathsf{c}_2(\mathsf{G}) > \mathrm{mr}(\mathsf{G}, \mathbb{F}_2)$, then $\mathrm{mr}(\mathsf{G}, \mathbb{F}_2)$ is even. Notice that

$$\mathrm{mr}(\mathsf{G} + \mathsf{K}_2, \mathbb{F}_2) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + \mathrm{mr}(\mathsf{K}_2, \mathbb{F}_2) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1,$$

which is odd.

By the lemma, $\mathsf{c}_2(\mathsf{G} + \mathsf{K}_2) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$. Since $\mathsf{c}_2(\mathsf{G}) \leqslant \mathsf{c}_2(\mathsf{G} + \mathsf{K}_2)$, we have $\mathsf{c}_2(\mathsf{G}) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$. □

Another upper bound

Let $\tau(G)$ denote the minimum cardinality of a vertex cover of G, *i.e.*, a subset of vertices U such that every edge of G is adjacent to a vertex in U.

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Idea of proof. Given a vertex cover U of G of size τ, we can obtain a collection of τ stars in G which partition the edges of G. Each star has $c_2(G) \leqslant 2$, from which the result follows.

Forests

Theorem (BPR)

For any n*-vertex forest* F*,*

$$\mathsf{c}_2(\mathsf{F}) = \mathrm{mr}(\mathsf{F}, \mathbb{F}_2) = \mathsf{n} - \mathsf{p}(\mathsf{F}),$$

where $\mathsf{p}(\mathsf{F})$ *is the minimum number of vertex-disjoint paths which cover all of the vertices of* F*.*

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The minimum rank problem is solved for forests:

Lemma (Chenette et al., 2007)

The minimum rank of a forest is independent of the field.

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Lemma (Johnson, Duarte, 1999)

For any n*-vertex tree* T*,* $\mathrm{mr}(\mathsf{T}, \mathbb{R}) = \mathsf{n} - \mathsf{p}(\mathsf{T})$*.*

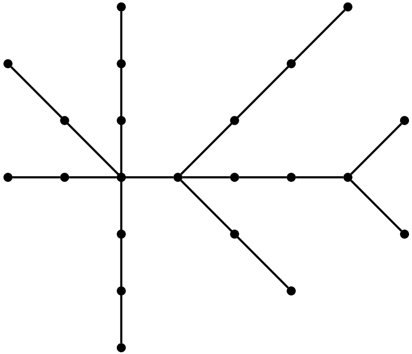
Forests

It suffices to show that $c_2(T) = n - p(T)$ for an n-vertex tree T.

Fallat and Hogben (2007) present an algorithm for finding a minimum collection of paths which covers $V(T)$:

- If T is a spider, take a maximal path through the center vertex, and all remaining paths.
- Otherwise, select pendant spiders one at a time, applying the last step.

A minimum path cover of a tree



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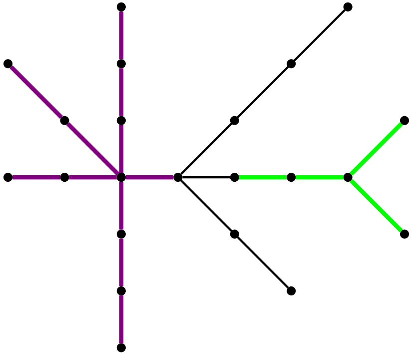

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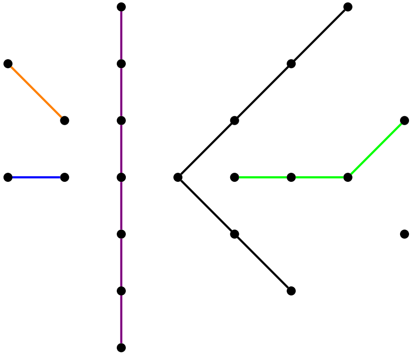

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where $\mathsf{p}(\mathsf{F})$ *is the minimum number of vertex-disjoint paths which cover all of the vertices of* F.

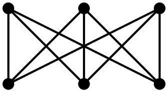
Proof.

- Use the algorithm to find a minimum path cover $\mathcal{P}$.
- The number of edges in $\mathcal{P}$ is $\mathsf{n} - \mathsf{p}(\mathsf{T})$.
- Each vertex of degree $\geqslant 3$ is not an endpoint of its path in $\mathcal{P}$, and thus sees two edges in $\mathcal{P}$.

With $\mathsf{n} - \mathsf{p}(\mathsf{G})$ subgraph complementations, we build the edges incident to high-degree vertices in 2 steps (as with stars), and all other edges one-by-one. □

Graphs with $\mathsf{c}_2(\mathsf{G}) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$

Example 1



Graphs with $c_2(G) = mr(G, \mathbb{F}_2) + 1$

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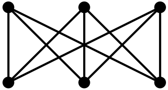
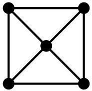
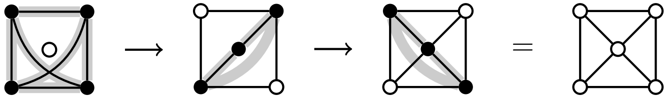


Example 2



A subgraph complementation system of W_5



Graphs with $c_2(G) = \mathrm{mr}(G, \mathbb{F}_2) + 1$

Lemma (BPR)

If $\mathscr{C}$ *is a subgraph complementation system for* G *in which every vetex is in an even number of complementations, then for all* $v \in V$*,*

$$\mathscr{C}_v = \{C \triangle \{v\} \mid C \in \mathscr{C}\}$$

is also a subgraph complementation system of G*.*

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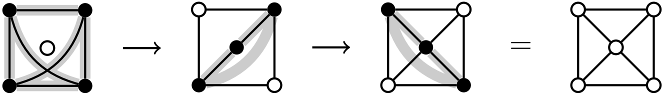

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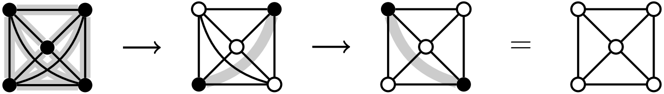

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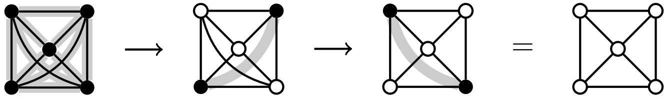

Corollary

If $\mathscr{C}$ is a complementation system of a graph G with $c_2(G)$ even, then some vertex of G is in an odd number of complementations in $\mathscr{C}$.

Graphs with $\mathsf{c}_2(\mathsf{G}) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$

Theorem (BPR)

Let G *be a nonempty graph. The following are equivalent.*

i. $\mathsf{c}_2(\mathsf{G}) \neq \mathrm{mr}(\mathsf{G}, \mathbb{F}_2)$*;*

ii. $\mathsf{c}_2(\mathsf{G}) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$*;*

iii. the adjacency matrix of G *is the only matrix which fits* G *and has minimum rank over* $\mathbb{F}_2$*;*

iv. there is a minimum subgraph complementation system for G *in which every vertex appears an even number of times;*

v. for every component G' *of* G, $\mathsf{c}_2(\mathsf{G}') = \mathrm{mr}(\mathsf{G}', \mathbb{F}_2) + 1$.

Graphs with $\mathsf{c}_2(\mathsf{G}) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$

Recall: if A is a rank k matrix over $\mathbb{F}_2$ which fits G, then there exists a matrix $X \in \mathbb{F}_2^{n \times k}$ such that either

- $A = XX^T$, or $X = M(\mathscr{C})$
- $A = A(G) = X \left(\oplus_1^l \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \right) X^T$, where $k = 2l$. ?

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For any graph G, *we have*

$$\mathrm{mr}(G, \mathbb{F}_2) = \min\{c_2(G), 2t_2(G)\},$$

where $t_2(G)$ *is the minimum number of tripartite subgraph complementations required to obtain* G *from the empty graph.*

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Pair the columns of X, $(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$, and let T_i be the $n \times 2$ matrix (A_i, B_i).

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Pair the columns of X, $(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$, and let T_i be the $n \times 2$ matrix (A_i, B_i).

To each T_i associate a complete tripartite graph with partite sets $(1, 0)$, $(0, 1)$, and $(1, 1)$.

Any vertex in G whose row in T_i is $(0, 0)$ is not in the triclique, all others are in their corresponding partite sets.

Graphs with $c_2(G) = \mathrm{mr}(G, \mathbb{F}_2) + 1$

Recall: if A is a rank k matrix over $\mathbb{F}_2$ which fits G, then there exists a matrix $X \in \mathbb{F}_2^{n \times k}$ such that either

- $A = XX^T$, or $X = M(\mathscr{C})$
- $A = A(G) = X\left(\oplus_1^l \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right) X^T$, where $k = 2l$. ?

Pair the columns of X, $(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$, and let T_i be the $n \times 2$ matrix (A_i, B_i).

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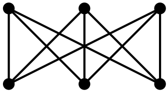
E is the symmetric difference of the edges of these l complete tripartite graphs, called a *tripartite subgraph complementation system* of G.

Graphs with $\mathsf{c}_2(\mathsf{G}) = \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) + 1$

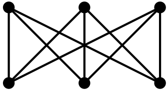
Example 1



$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

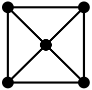
Graphs with $c_2(G) = \mathrm{mr}(G, \mathbb{F}_2) + 1$

Example 1



$$\begin{pmatrix}1 & 0\\1 & 0\\1 & 0\\0 & 1\\0 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 1 & 1\end{pmatrix}$$

Example 2



$$\begin{pmatrix}1 & 1\\1 & 0\\0 & 1\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 0 & 1 & 0\\1 & 0 & 1 & 0 & 1\end{pmatrix}$$

Forbidden induced subgraphs

Theorem (BPR)

The class of graphs with $\mathsf{c}_2(\mathsf{G}) \leqslant \mathsf{k}$ *is defined by a finite set of minimal forbidden induced subgraphs.*

Forbidden induced subgraphs

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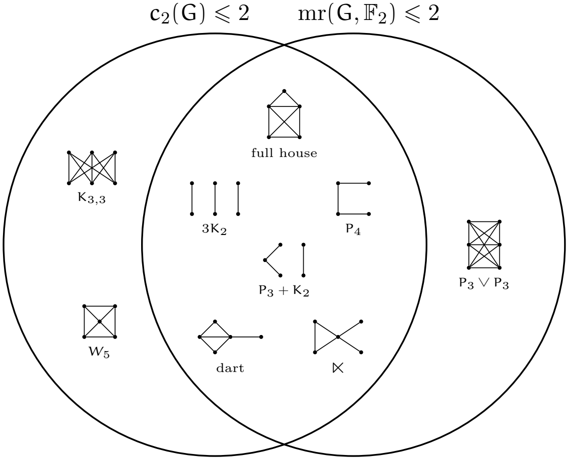
The class of graphs with $\mathsf{c}_2(\mathsf{G}) \leqslant \mathsf{k}$ *is defined by a finite set of minimal forbidden induced subgraphs.*

For odd k,

$$\{\mathsf{G} \mid \mathsf{c}_2(\mathsf{G}) \leqslant \mathsf{k}\} = \{\mathsf{G} \mid \mathrm{mr}(\mathsf{G}, \mathbb{F}_2) \leqslant \mathsf{k}\}.$$

In particular, the sets of minimal forbidden induced subgraphs for these classes are equal.

Minimal forbidden induced subgraphs



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