Saturation & Rainbow Saturation Numbers of Certain Trees

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University of Vermont

AMS Spring Eastern Sectional Meeting; Hartford, CT

Special Session on Recent Trends on Graphs & Hypergraphs

April 5, 2025

Saturation: joint w/Puck Rombach

Rainbow: w/Neal Bushaw, Daniel P. Johnston, & Puck Rombach

Plan

Introduction to (semi)saturation

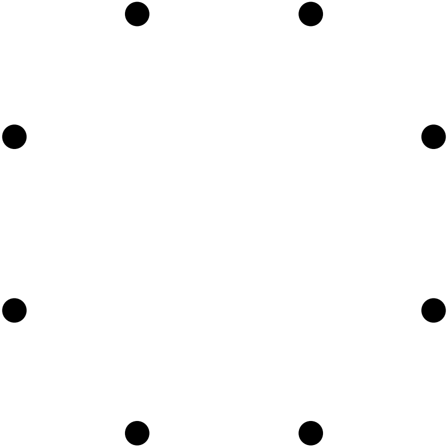
Lower bounds on semisaturation

Double stars

Graph saturation

Add edges, avoiding a forbidden graph H, until you're stuck

Example ($H = K_3$)



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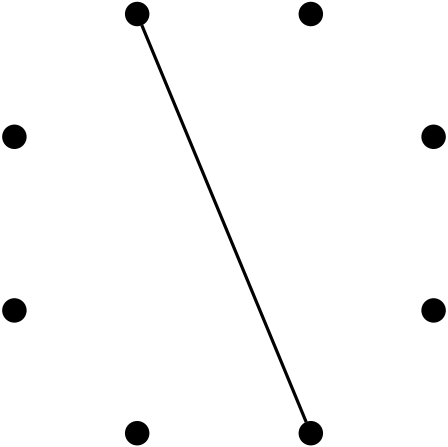

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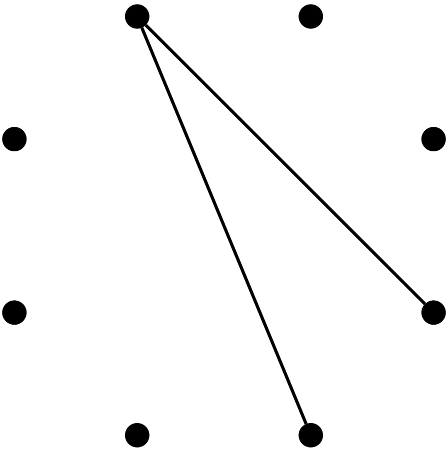

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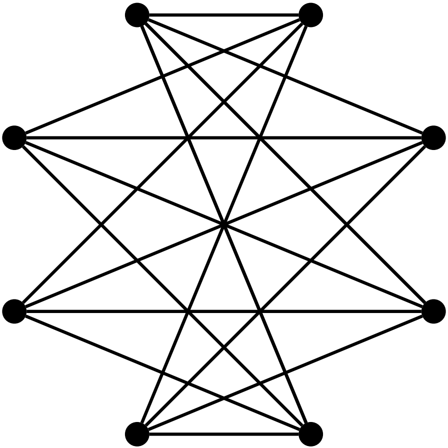

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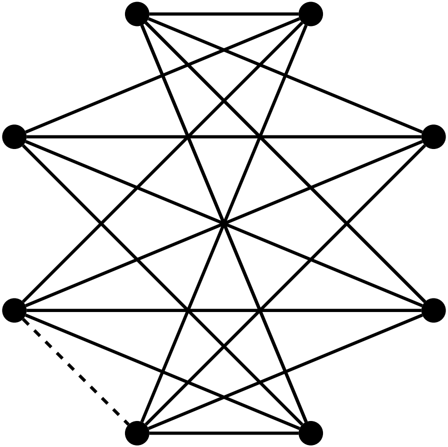

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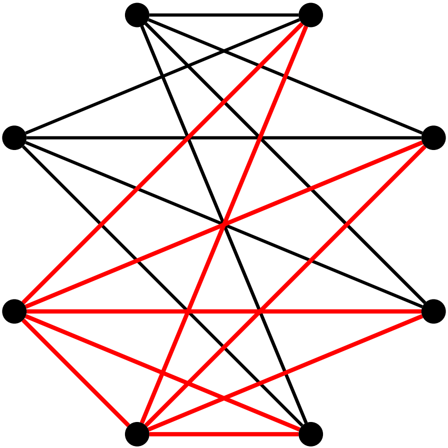

Extremal numbers

Theorem ([Mantel 1907])

The maximum size of a K_3-free graph of order n is $\lfloor n^2/4 \rfloor$.

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Theorem ([Turán 1941])

The maximum size of a K_{p+1}-free graph of order n is

$$\left(1 - \frac{1}{p}\right)\frac{n^2}{2} - \frac{s(p-s)}{2p},$$

where s is the remainder of n/p. Further, this is witnessed by a unique graph for every n.

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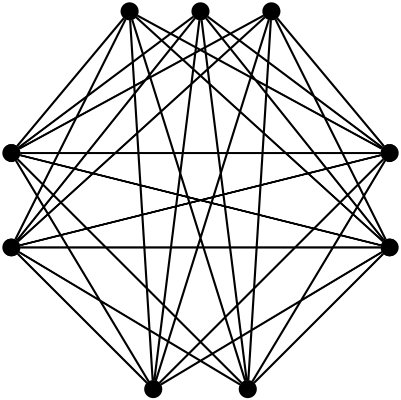

The unique graph of maximum size over all K_5-free graphs of order 9

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Theorem ([Erdős-Stone 1946, Erdős-Simonovits 1966])

The maximum size of an H-free graph of order n is

$$\left(1 - \frac{1}{\chi(H) - 1}\right)\frac{n^2}{2} + o(n^2).$$

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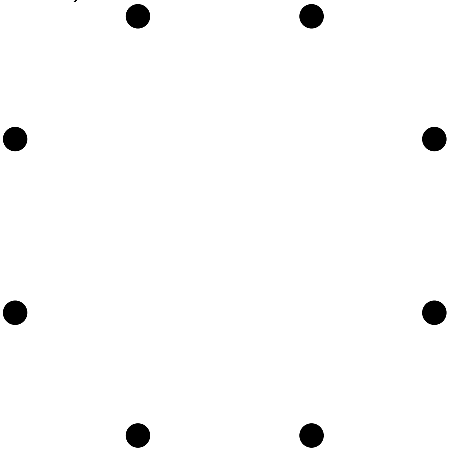

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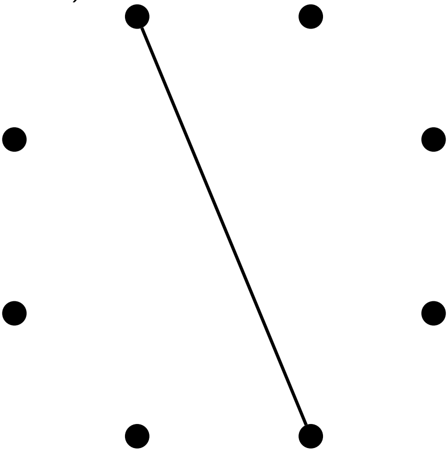

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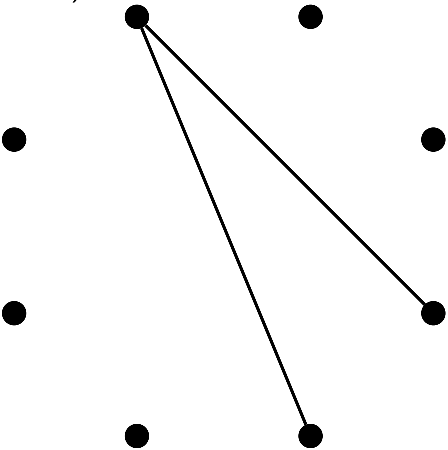

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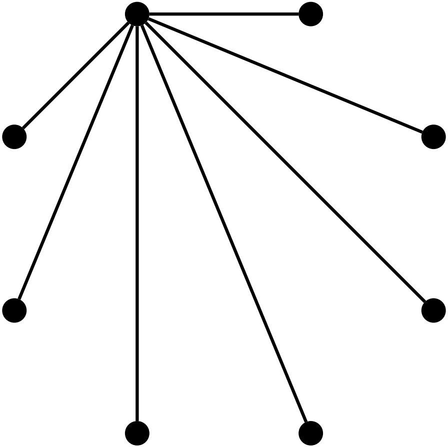

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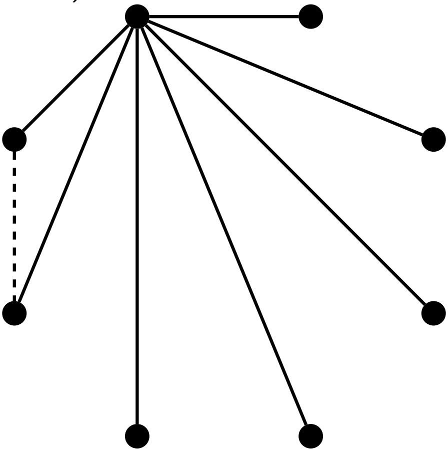

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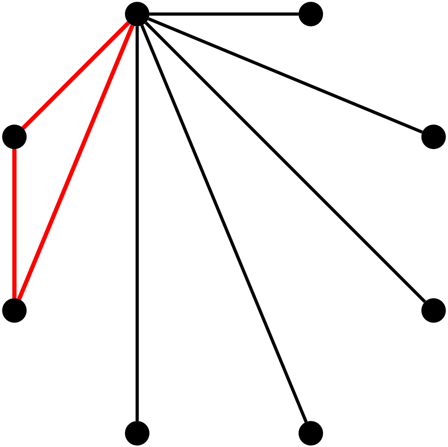

Saturation numbers (maximum $\to$ maximal)

G is *H-saturated* if

- G is H-free, and
- the addition of any extra edge to G creates a copy of H.

The *saturation number* $\mathrm{sat}(n, H)$ is the minimum size of an H-saturated graph of order n.

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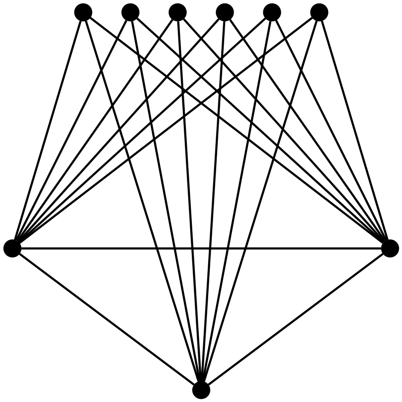
Theorem ([Erdős-Hajnal-Moon 1964])

$\mathrm{sat}(n, K_{p+1}) = (p-1)(n-p+1) + \binom{p-1}{2}$, *and this is witnessed by a unique graph for every n.*

The graph of Erdős, Hajnal, and Moon



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Semisaturation numbers

G is *H-semisaturated* if

- ~~G is H-free, and~~
- the addition of any extra edge to G creates a copy of H.

The *semisaturation number* $\mathrm{ssat}(n, H)$ is the minimum size of an H-semisaturated graph of order n.

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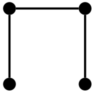
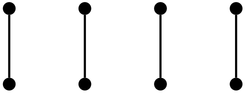
Saturation and semisaturation numbers

Example (P_4)

$H =$ 

$G =$ 

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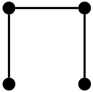

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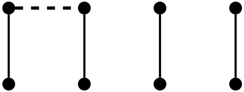

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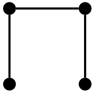
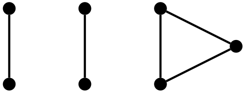
$H =$ 

$G =$ 

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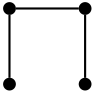
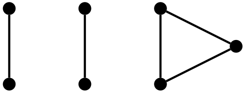
$H =$  $G =$ 

Since G is P_4-saturated, $\mathrm{ssat}(n, P_4) \leqslant \mathrm{sat}(n, P_4) \leqslant \lceil n/2 \rceil + 1$.

For any graph H without an isolated edge, $\mathrm{ssat}(n, P_4) \geqslant \lfloor n/2 \rfloor$.

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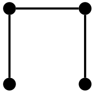
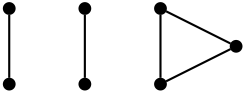
$H =$ 

$G =$ 

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- $\mathrm{sat}(n, H) = O(n)$

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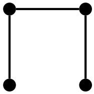
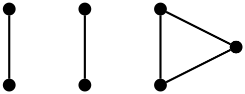
$H =$  $G =$ 

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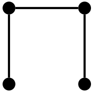
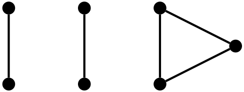
$H =$  $G =$ 

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$$a_\ell = \begin{cases} 3 \cdot 2^{m-1} - 2 : & \ell = 2m \\ 4 \cdot 2^{m-1} - 2 : & \ell = 2m+1 \end{cases}$$

Saturation and semisaturation numbers

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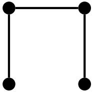
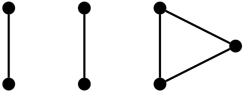
$H =$ 

$G =$ 

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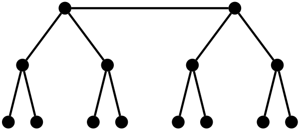
$T^2_6 =$ 

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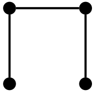
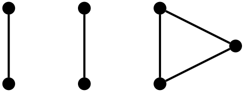
$H =$ 

$G =$ 

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[Burr 2017]: $\text{ssat}(n, P_\ell) = n - \lfloor n/b_\ell \rfloor + O(1)$, where
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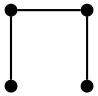
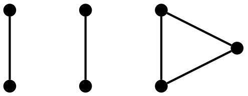
$H =$ 

$G =$ 

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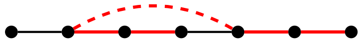

Trees [Faudree-Faudree-Gould-Jacobson 2009]

Theorem

Let $T \neq K_{1,p-1}$ be a tree of order $p \geqslant 5$ with second smallest degree δ_2. If $n \geqslant (d-1)^3$, then $\mathrm{sat}(n, T) \geqslant (\delta_2 - 1)n/2$.

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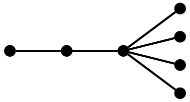
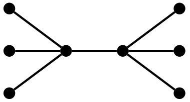
Let $S_{s,t}$ denote the double star obtained by joining the centers of $K_{1,s-1}$ and $K_{1,t-1}$.

$S_{2,5} =$ 

$S_{4,4} =$ 

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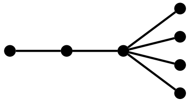
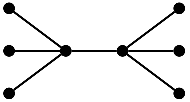
$S_{2,5} =$  $S_{4,4} =$ 

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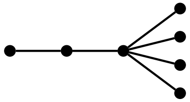
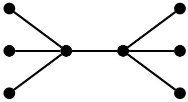
$S_{2,5} =$ 

$S_{4,4} =$ 

- $\mathrm{sat}(n, S_{2,p-2}) = n - \lfloor (n+p-2)/p \rfloor$, which is minimum over all trees of order p.
- For $n \geqslant s^3$ and $s \leqslant t$, $\mathrm{sat}(n, S_{s,s}) = (s-1)n/2 + O(1)$, and

$$\frac{s-1}{2} n \leqslant \mathrm{sat}(n, S_{s,t}) \leqslant \frac{s}{2} n + O(1).$$

Lower bounds on $\mathrm{ssat}(n, H)$

For each edge uv in a graph H, define

$$\mathrm{wt}_0(uv) = \max\{d(u), d(v)\} - 1,$$

and let $k_0 = \min_{uv \in E(H)}\{\mathrm{wt}_0(uv)\}$.

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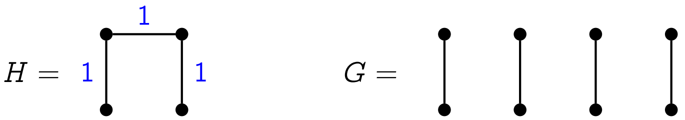

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Proposition

For any graph H and integer $n \geqslant |H|$,

$$\mathrm{ssat}(n, H) \geqslant k_0 \cdot \frac{n}{2} - \frac{(k_0+1)^2}{8}.$$

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Theorem ([Cameron-Puleo 2022])

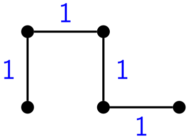
For any graph H and integer $n \geqslant |H|$,

$$\mathrm{ssat}(n, H) \geqslant w \cdot \frac{n}{2} - O(1),$$

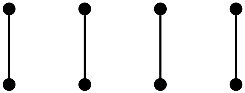
where $w = \min_{uv \in E(H)}\{\mathrm{wt}_0(uv) + |N(u) \cap N(v)|\}$.

Lower bounds on $\mathrm{ssat}(n, H)$

$P_5 =$

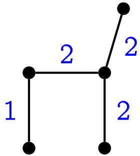


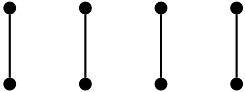
$G =$



not P_5-saturated

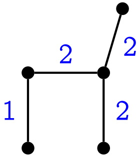
Lower bounds on $\mathrm{ssat}(n, H)$

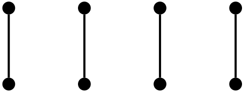
$S_{2,3} =$ 

$G =$ 

not $S_{2,3}$-saturated

Lower bounds on $\mathrm{ssat}(n, H)$

$S_{2,3} =$ 

$G =$ 

not $S_{2,3}$-saturated

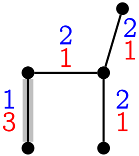
For each edge uv in a graph H, define

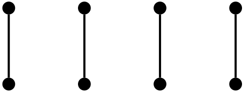
$$\mathrm{wt}_1(uv) = \max\{d(w) : w \in (N(u) - v) \cup (N(v) - u)\}.$$

Let

$$k_1 = \min_{uv \in E(H)} \{\mathrm{wt}_1(uv)\} \quad \text{and} \quad k_1' = \min_{\mathrm{wt}_0(uv) = k_0} \{\mathrm{wt}_1(uv)\}.$$

Lower bounds on $\mathrm{ssat}(n, H)$

$S_{2,3} =$ 

$G =$ 

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Lower bounds on $\mathrm{ssat}(n, H)$

Theorem ([Buchanan-Rombach 2024])

For any graph H and integer $n \geqslant |H|$,

$$\mathrm{ssat}(n, H) \geqslant \left(k_0 + \frac{k_1' - k_0}{k_1' + 1}\right)\frac{n}{2} - O(1). \tag{1}$$

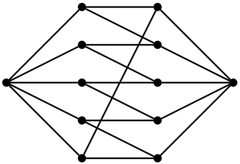
Further, if $k_1 > k_0$, then

$$\mathrm{ssat}(n, H) \geqslant \left(k_0 + \frac{k_1' - k_0}{k_1'}\right)\frac{n}{2} - O(1). \tag{2}$$

Lower bounds on ssat(n, H)

In other words, the average degree of an H-semisaturated graph cannot be much smaller than that of a graph with minimum degree k_0 in which

(1) every vertex of degree k_0 has a neighbor of degree k_1'.



$k_0 = 3$, $k_1' = 5$

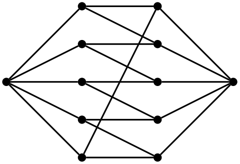
(1)

Lower bounds on ssat(n, H)

In other words, the average degree of an H-semisaturated graph cannot be much smaller than that of a graph with minimum degree k_0 in which

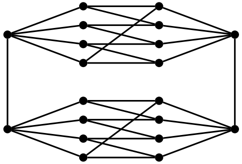
(1) every vertex of degree k_0 has a neighbor of degree k_1'.

(2) every vertex has a neighbor of degree k_1' (when $k_1 > k_0$).



(1)

$k_0 = 3,\ k_1' = 5$



(2)

Triangle-free graphs H

Theorem ([Buchanan-Rombach 2024])

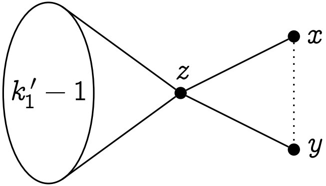
Let H be a triangle-free graph such that $k_1' \geqslant k_0 + \sqrt{2k_0+1}$, or at least one degree-(k_0+1) endpoint of every edge minimizing wt_0 has a neighbor of degree k_1' and $k_1' \geqslant k_0 + 2$. For any $n \geqslant |H|$,

$$\mathrm{ssat}(n, H) \geqslant \left(k_0 + \frac{k_1' + 1 - k_0}{k_1' + 2}\right)\frac{n}{2} - O(1). \tag{3}$$

If, in addition to either of the above conditions, $k_1 > k_0$, then

$$\mathrm{ssat}(n, H) \geqslant \left(k_0 + \frac{k_1' + 1 - k_0}{k_1' + 1}\right)\frac{n}{2} - O(1). \tag{4}$$

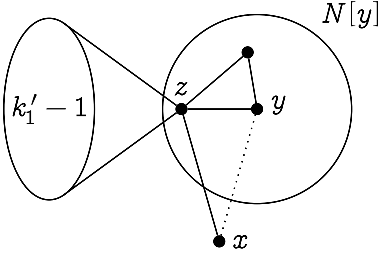
Triangle-free graphs H



Nonadjacent degree-k_0 vertices x, y in H-semisaturated graph

Triangle-free graphs H

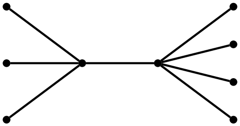
If every edge in H minimizing wt_0 has a degree-(k_0+1) endpoint with a neighbor of degree at least k_1':



Nonadjacent degree-k_0 vertices x, y in H-semisaturated graph

Double stars

Let $S_{s,t}$ be obtained by joining the centers of $K_{1,s-1}$ and $K_{1,t-1}$



The double star $S_{4,5}$

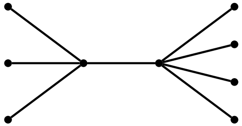
Theorem ([Faudree-Faudree-Gould-Jacobson 2009])

For any $2 \leqslant s \leqslant t$ *and* $n \geqslant s^3$,

$$\frac{s-1}{2}n \leqslant \mathrm{sat}(n, S_{s,s}) \leqslant \frac{s-1}{2}n + \frac{s^2-1}{2}, \quad \textit{and}$$
$$\frac{s-1}{2}n \leqslant \mathrm{sat}(n, S_{s,t}) \leqslant \frac{s}{2}n - \frac{(s-1)^2+8}{8}.$$

Double stars

Let $S_{s,t}$ be obtained by joining the centers of $K_{1,s-1}$ and $K_{1,t-1}$



The double star $S_{4,5}$

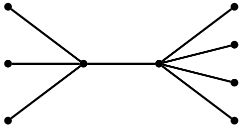
Theorem ([Buchanan-Rombach 2024])

For any $2 \leqslant s < t$ and $n \geqslant s + t$,

$$\text{ssat}(n, S_{s,t}) \geqslant \frac{s(t+1)n - s(t-s+2)}{2t+4} - \frac{s^2}{8}.$$

Double stars

Let $S_{s,t}$ be obtained by joining the centers of $K_{1,s-1}$ and $K_{1,t-1}$



The double star $S_{4,5}$

Theorem ([Buchanan-Rombach 2024])

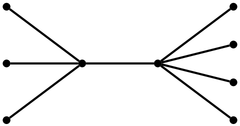
For any $2 \leqslant s < t$ *and* $n \geqslant q(2t+4)+s$,

$$\mathrm{sat}(n, S_{s,t}) \leqslant \frac{s(t+1)n + s(s-1)}{2t+4} + \left\lceil \frac{s}{2} \right\rceil,$$

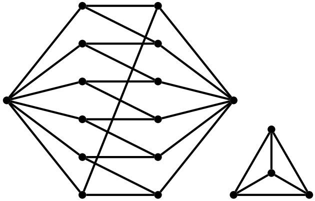
where $q = \max\{1, \lfloor s/2 \rfloor - 1\}$.

Double stars

Let $S_{s,t}$ be obtained by joining the centers of $K_{1,s-1}$ and $K_{1,t-1}$



The double star $S_{4,5}$



An $S_{4,5}$-saturated graph

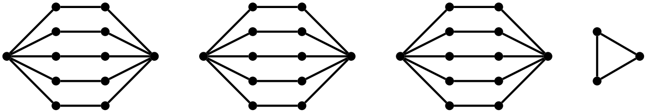
Double stars

Theorem ([Buchanan-Rombach 2024])

For any $2 \leqslant s < t$, there exists $n_0 = n_0(s,t)$ such that, for all $n \geqslant n_0$,

$$\mathrm{ssat}(n, S_{s,t}) \geqslant \frac{s(t+1)n - s(t-s+2)}{2t+4},$$

and this is sharp when $n \equiv s \pmod{2t+4}$.



A graph of minimum size over all $S_{3,4}$-(semi)saturated graphs of order 39

(proper) Rainbow saturation

An edge coloring of a graph is

- *proper* if incident edges have different colors;
- *rainbow* if all edges have different colors;
- *rainbow H-free* if no H subgraph is rainbow.

(proper) Rainbow saturation

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A graph is *rainbow H-saturated* if it is edge-maximal w.r.t. having a rainbow H-free proper edge coloring.

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$\mathrm{ex}^\star(n, H) =$ maximum size of a rainbow H-saturated graph of order n

Theorem ([Keevash-Mubayi-Sudakov-Verstraëte 2007])
$\mathrm{ex}^\star(n, H) \approx \mathrm{ex}(n, H)$ *when* $\chi(H) \geqslant 3$

(proper) Rainbow saturation

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A graph is *rainbow H-saturated* if it is edge-maximal w.r.t. having a rainbow H-free proper edge coloring.

$\mathrm{sat}^{\star}(n, H)$ = minimum size of a rainbow H-saturated graph of order n

Theorem ([Bushaw-Johnston-Rombach 2022])

$\mathrm{sat}^{\star}(n, H) = O(n)$ *when H contains no induced even cycle.*

(proper) Rainbow saturation

An edge coloring of a graph is

- *proper* if incident edges have different colors;
- *rainbow* if all edges have different colors;
- *rainbow H-free* if no H subgraph is rainbow.

A graph is *rainbow H-saturated* if it is edge-maximal w.r.t. having a rainbow H-free proper edge coloring.

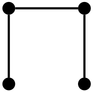
$\mathrm{sat}^\star(n, H)$ = minimum size of a rainbow H-saturated graph of order n

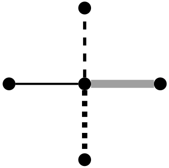
Theorem ([Various sources])

$\mathrm{sat}^\star(n, H) = O(n)$ *for any graph H.*

(proper) Rainbow saturation

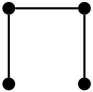
Example (P_4)

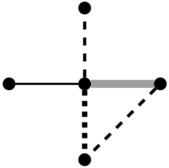
$H =$ 

$G =$ 

(proper) Rainbow saturation

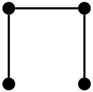
Example (P_4)

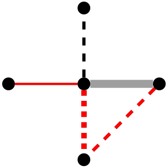
$H =$ 

$G =$ 

(proper) Rainbow saturation

Example (P_4)

$H =$ 

$G =$ 

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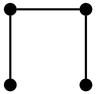
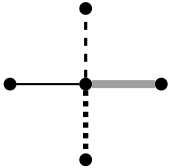
$H =$ 

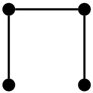
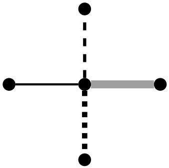
$G =$ 

Theorem ([Bushaw-Johnston-Rombach 2022])

$$\mathrm{sat}^{\star}(n, P_4) = \frac{4}{5}n + O(1)$$

(proper) Rainbow saturation

Example (P_4)

$H =$  $G =$ 

Theorem ([Lane-Morrison 2024])

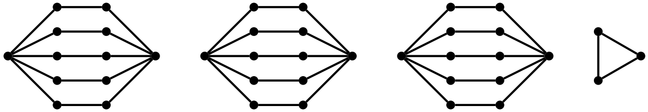
$$\mathrm{sat}^{\star}(n, S_{2,t}) = n - \left\lfloor \frac{n+t+1}{t+3} \right\rfloor = \frac{t+2}{t+3}n + O(1)$$

Double stars

Theorem ([Buchanan-Rombach 2024])

For any $2 \leqslant s < t$, there exists $n_0 = n_0(s,t)$ such that, for all $n \geqslant n_0$,

$$\mathrm{ssat}(n, S_{s,t}) \geqslant s\left(\frac{t+1}{t+2}\right)\frac{n}{2} - \frac{s(t-s+2)}{2t+4}.$$



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Note that $\mathrm{sat}^\star(n,H) \geqslant \mathrm{ssat}(n,H)$.

Corollary

For any $2 \leqslant s < t$, there exists $n_0 = n_0(s,t)$ such that, for all $n \geqslant n_0$,

$$\mathrm{sat}^\star(n, S_{s,t}) \geqslant s\left(\frac{t+1}{t+2}\right)\frac{n}{2} - \frac{s(t-s+2)}{2t+4}.$$

Double stars

Theorem ([Buchanan-Bushaw-Johnston-Rombach 2025^+])

For any $2 \leqslant s \leqslant t$,

$$\mathrm{sat}^\star(n, S_{s,t}) \leqslant s\left(\frac{s+t}{s+t+1}\right)\frac{n}{2} + O(1).$$

Double stars

Theorem ([Buchanan-Bushaw-Johnston-Rombach 2025^{+}])

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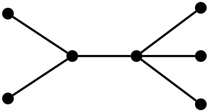
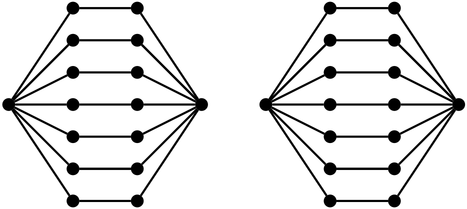


$S_{3,4}$



G

Double stars

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For any $2 \leqslant s \leqslant t$,

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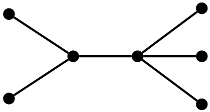

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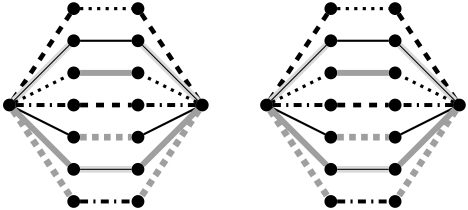
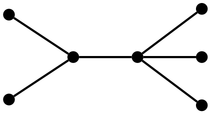


G

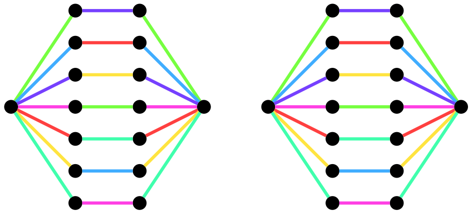
Double stars

Theorem ([Buchanan-Bushaw-Johnston-Rombach 2025⁺])
For any $2 \leqslant s \leqslant t$,

$$\mathrm{sat}^\star(n, S_{s,t}) \leqslant s\left(\frac{s+t}{s+t+1}\right)\frac{n}{2} + O(1).$$



$S_{3,4}$



G

Double stars

Theorem ([Buchanan-Bushaw-Johnston-Rombach 2025$^+$])
For any $2 \leqslant s \leqslant t$,

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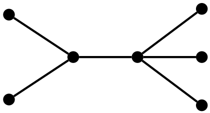
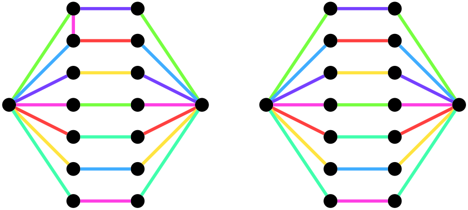


$S_{3,4}$



G

Double stars

Theorem ([Buchanan-Bushaw-Johnston-Rombach 2025^+])

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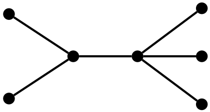
$$\mathrm{sat}^{\star}(n, S_{s,t}) \leqslant s\left(\frac{s+t}{s+t+1}\right)\frac{n}{2} + O(1).$$



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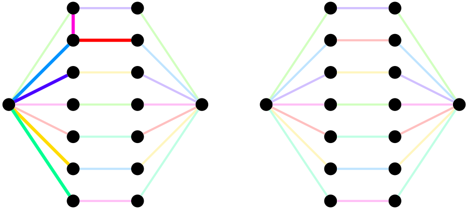

G

Thank you!

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