On the last new vertex visited by a random walk in a directed graph

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Cover tours

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Cycles and complete graphs have the property that a random cover tour, starting at any vertex, is equally likely to end at any other vertex.

Ronald Graham asked if there are any other such graphs.

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Lemma (Lovász-Winkler, 1993)

If G is connected and $uv \notin E(G)$, then there is a neighbor x of u such that $\mathbb{P}(L(x, v)) \leqslant \mathbb{P}(L(u, v))$. Further, this inequality is strict if there is a cover tour of G from u to v which does not revisit u.

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Theorem (Lovász-Winkler, 1993)

Cycles and complete graphs are the only undirected graphs with the property that $\mathbb{P}(L(u, v)) = \mathbb{P}(L(u, w))$ for any three distinct vertices u, v, and w.

Directed graphs

We denote by $L(u, v)$ the event that v is the last vertex visited by a random cover tour of a digraph G starting at vertex u.

Lemma (B.-Horn-Rombach, 2023)

If G is strongly connected and $uv \notin E(G)$, then there is an out-neighbor x of u such that $\mathbb{P}(L(x, v)) \leqslant \mathbb{P}(L(u, v))$. Further, this inequality is strict if there is a cover tour from u to v which does not revisit u.

Theorem (B.-Horn-Rombach, 2023)

Cycles and complete graphs are the only directed graphs with the property that $\mathbb{P}(L(u, v)) = \mathbb{P}(L(u, w))$ for any three distinct vertices u, v, and w.*

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*with all edges considered bidirected

Proof of theorem

It suffices to show that, in any digraph with the property that $\mathbb{P}(L(u,v)) = \mathbb{P}(L(u,w))$ for any three distinct vertices u, v, and w, every edge is bidirected.

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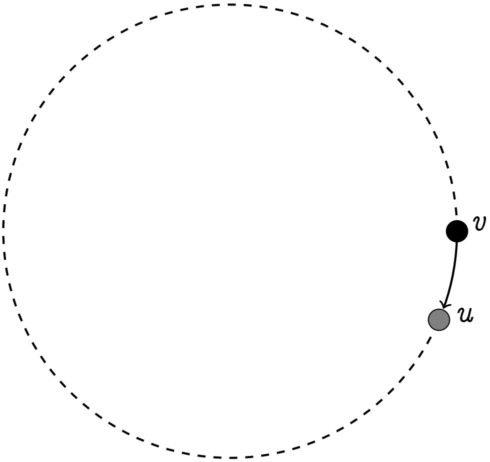
By our lemma, if G is a digraph with the above property, and if T is a cover tour in G from u to v, then either $uv \in E(G)$ or u appears at least twice in T.

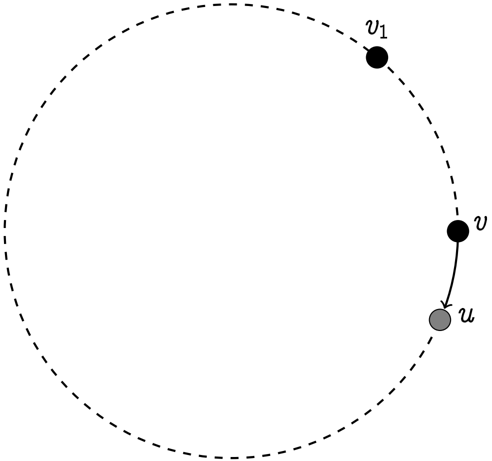
Proof of theorem

Suppose, for a contradiction, that $uv \notin E(G)$ but $vu \in E(G)$. Consider a cover tour T from u to v of minimum length.



Proof of theorem

Let v_1 be the last new vertex visited by the walk which first takes vu then follows T.



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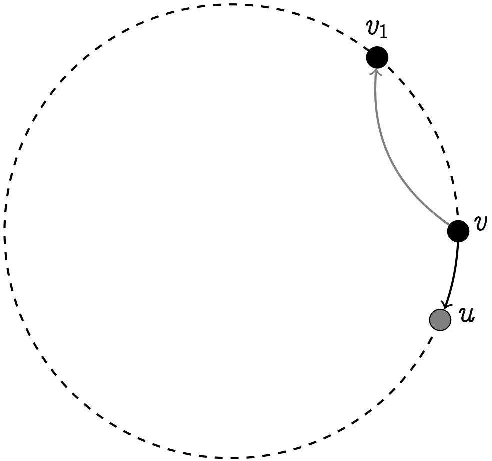
Minimality of $T \implies v_1$ appears only once $\implies vv_1 \in E(G)$.



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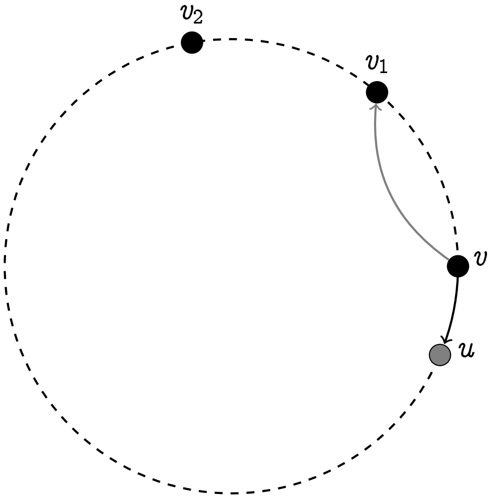

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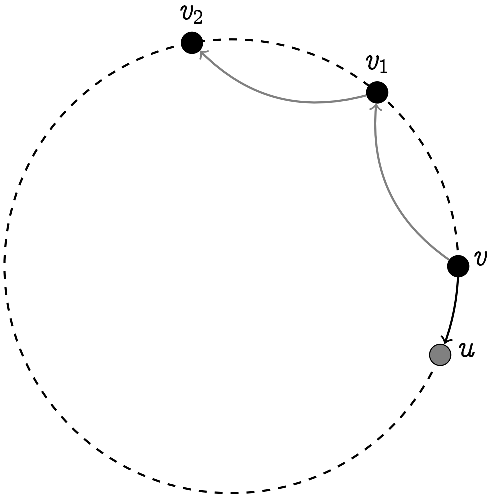

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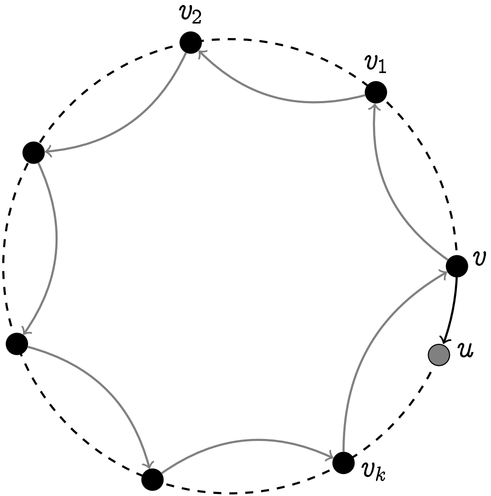
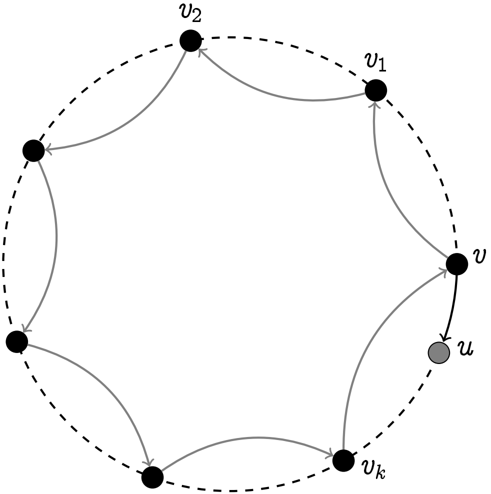


Proof of theorem

We now show that $v_1 v \in E(G)$ by finding a cover tour from v_1 to v which visits v_1 only once.



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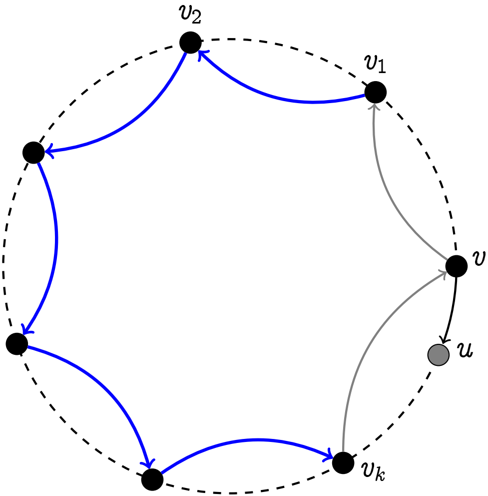
Part (I) starts at v_1



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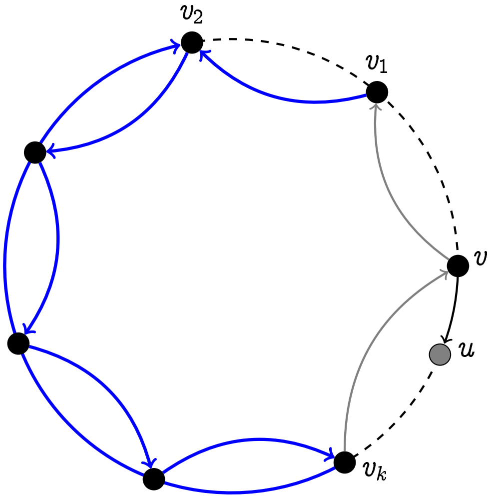
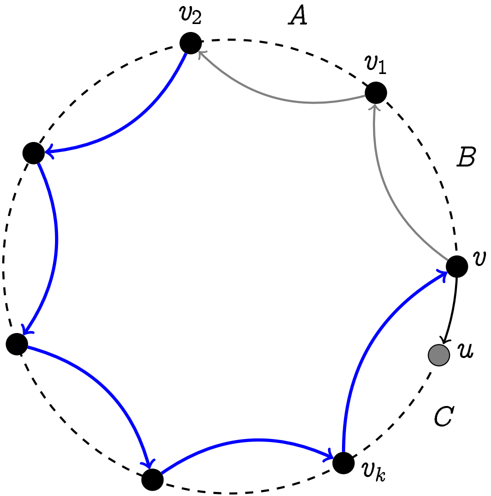
We now show that $v_1 v \in E(G)$ by finding a cover tour from v_1 to v which visits v_1 only once.

Part (I) starts at v_1 and stops at v_2.



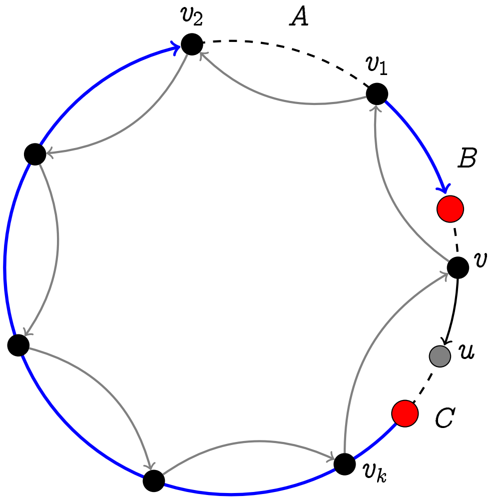
Proof of theorem

Part (II) starts at v_2 and ends the cover tour at v.



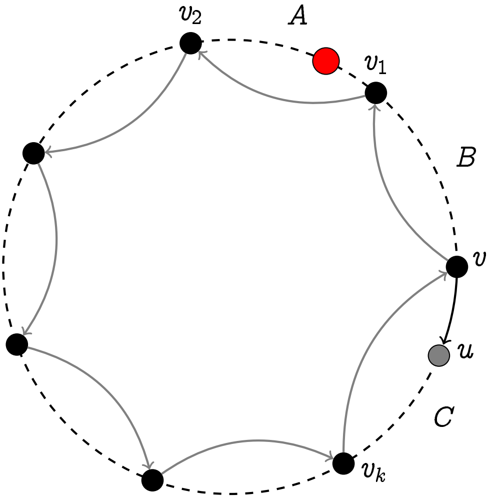
Proof of theorem

If there is an unseen vertex in B and C, Part (I) becomes:



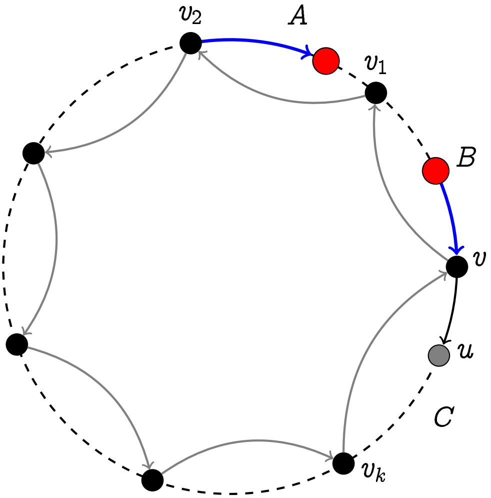
Proof of theorem

Any remaining unseen vertices have one copy in A and another in B or C. Let y be the last unseen vertex in A.



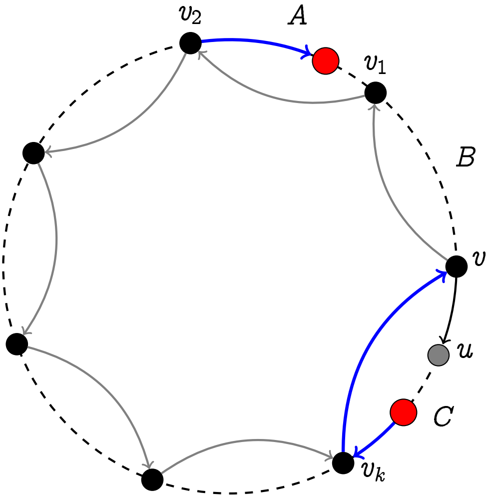
Proof of theorem

If y has a copy in B, Part (II) becomes:



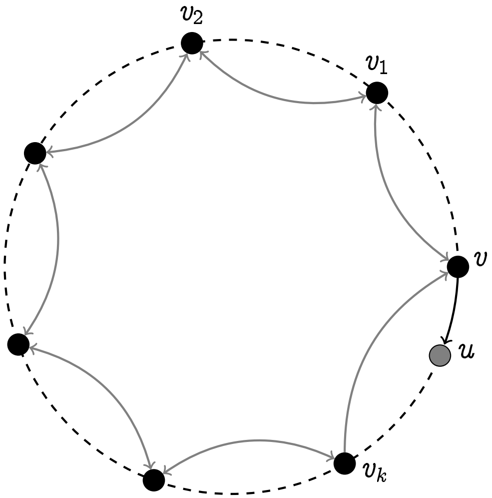
Proof of theorem

Otherwise, y has a copy in C. Part (II) becomes:



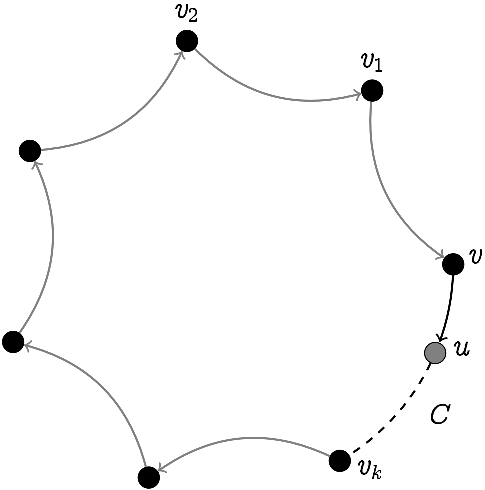
Proof of theorem

Similarly, we have $v_{i+1}v_i \in E(G)$ for each $i \in \{1, \ldots, k\}$.



Proof of theorem

In fact, the edge $v_1 v$ and each $v_{i+1} v_i$ is in T, by minimality.
But u appears twice in T, and not twice in C, a contradiction.



Thank you!

References

C. Buchanan, P. Horn, P. Rombach, On the last new vertex visited by a random walk in a directed graph, *Discrete Math. Lett.*, Vol. 11, (2023) 96-98.

L. Lovász, P. Winkler, A note on the last new vertex visited by a random walk, *J. Graph Theory*, Vol. 17, No. 5, (1993) 593-596.