Graduate Research Workshop in Combinatorics

Odd Covers of Graphs

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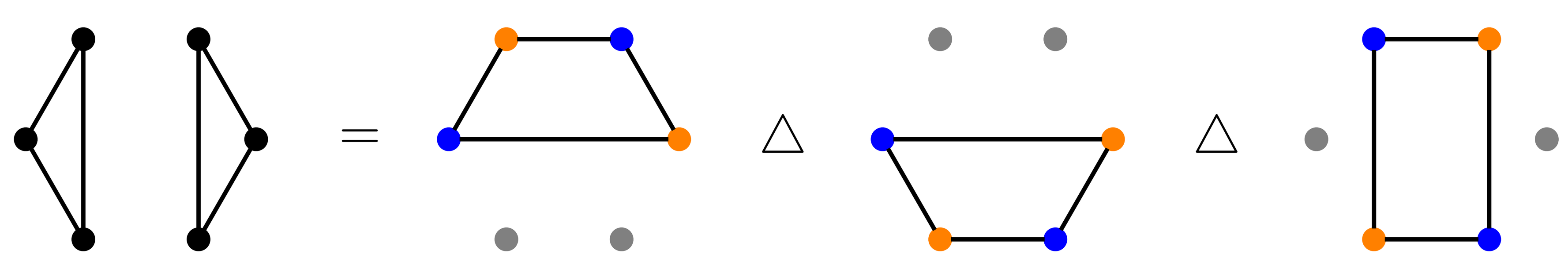
Odd covers and $b_2(G)$

Let $G = (V, E)$ be a simple graph. An *odd cover* of G is a collection of bicliques (complete bipartite graphs) on subsets of V with the property that $uv \in E$ if and only if uv is in an odd number of bicliques. (The corresponding problem when the graphs in an odd cover are cliques is studied in [3].)

Let $b_2(G)$ denote the minimum cardinality of an odd cover of G. For example, we have $b_2(2K_3) = 3$; a minimum odd cover of $2K_3$ is depicted below.



General lower bound ([2]). *For any graph G,*

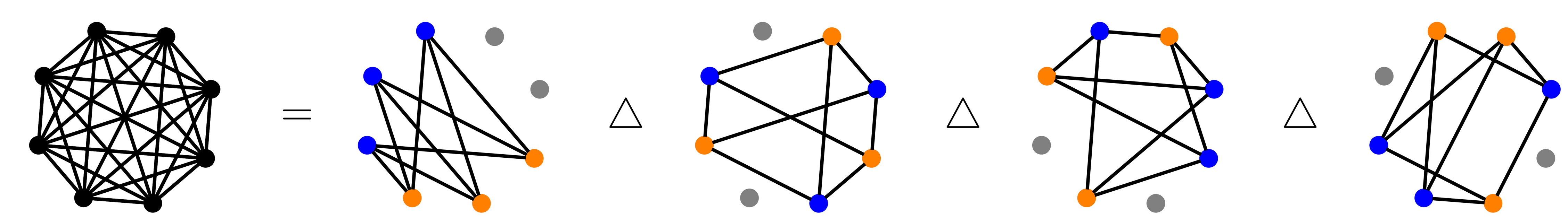
$$2b_2(G) \geq \operatorname{rank}_{\mathbb{F}_2}(A(G)).$$

Idea of proof. First, if $G_1, \ldots, G_k$ are bicliques (along with isolated vertices) which form an odd cover of G, then $A(G) = \sum A(G_i) \mod 2$. Second, matrix rank is subadditive.

Complete graphs

The "odd cover problem," a variation of the Graham-Pollak problem, was posed by Babai and Frankl [1]: *What is the minimum number of bicliques which cover every edge of K_n an odd number of times?*

The authors of [5] noticed the following odd cover of K_8 and solved the odd cover problem for an infinite but density-0 set of integers.



Theorem ([2]). *For any positive integer n,*

$$\left\lceil \frac{n}{2} \right\rceil \leq b_2(K_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1.$$

In particular, $b_2(K_n) = \lceil n/2 \rceil$ when $8 \mid n$ or $n \equiv \pm 1 \mod 8$.

Conjecture ([2]). *For $n \geq 2$, $b_2(K_{2n}) = n$ if and only if $4 \mid n$, and $b_2(K_{2n-1}) = n$.*

Bipartite graphs

Theorem ([2]). *If G is bipartite, then*

$$2b_2(G) = \operatorname{rank}_{\mathbb{F}_2}(A(G)).$$

Furthermore, there exists a minimum odd cover of G that respects its bipartition.

Let $\tau(G)$ be the vertex cover number of G. For any forest F, it is known that $\operatorname{rank}_{\mathbb{F}_2}(A(F)) = \tau(F)$ [4].

Corollary ([2]). *For any forest F, we have $b_2(F) = \tau(F)$.*

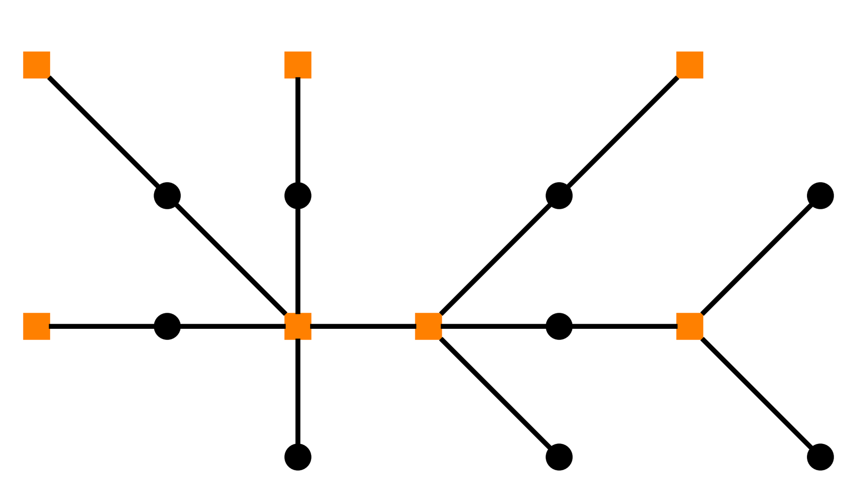


Figure: A minimum vertex cover of a forest (in orange) induces a minimum odd cover.

Corollary ([2]). *For $n \geq 2$, $b_2(C_{2n}) = n - 1$.*

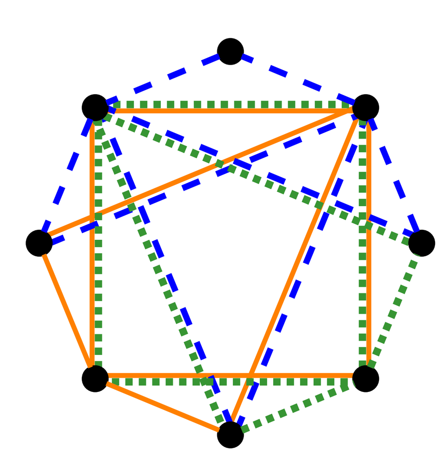
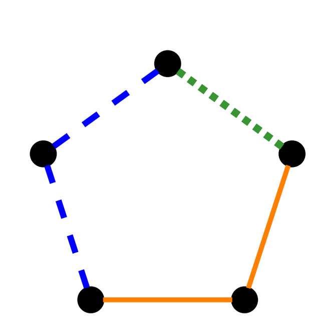


Figure: A minimum odd cover of C_8. A construction for C_{2n} is left as an exercise.

Odd cycles

Theorem ([2]). *For $n \geq 2$, $b_2(C_{2n-1}) = n$.*

Unlike C_{2n}, minimum biclique partition of C_{2n+1} is a minimum odd cover.



An upper bound and T_k

For any graph G with $\operatorname{rank}_{\mathbb{F}_2}(A(G)) = 2k$, there exists an $n \times 2k$ matrix M such that

$$A(G) = M \left(\oplus_1^k \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \right) M^T. \qquad (1)$$

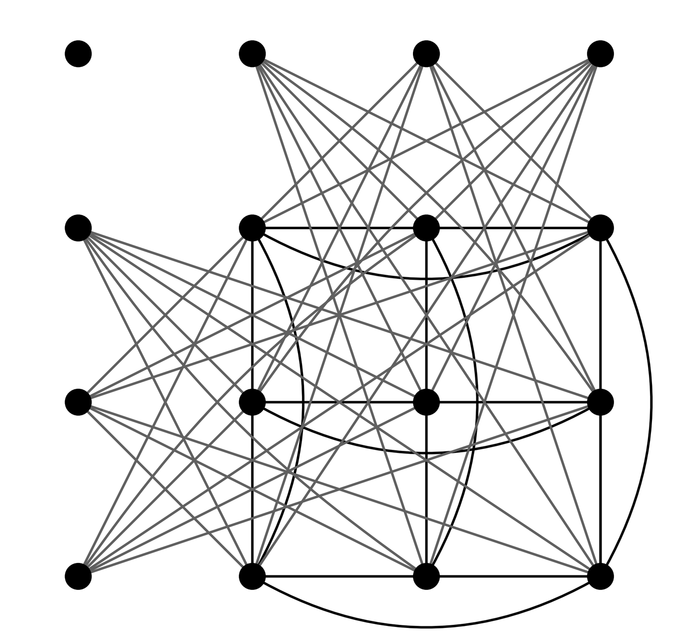
There is a collection of k tricliques associated to M whose symmetric difference of edge sets is $E(G)$. Since each triclique can replaced by two bicliques, we obtain the following.

General upper bound ([2]).

$$b_2(G) \leq \operatorname{rank}_{\mathbb{F}_2}(A(G)).$$

The graph T_k, whose adjacency matrix is given by (1) when M is the $n \times 2^{2k}$ matrix whose rows are all distinct vectors over $\mathbb{F}_2$ of length $2k$, is well-known.

$T_2 =$



The rank over $\mathbb{F}_2$ of $A(T_k)$ is $2k$, and our general upper bound is tight for T_1 and T_2. For larger k, we have

$$b_2(T_k) \geq \log_3(4) \cdot k,$$

but we conjecture $b_2(T_k) = 2k$ for any k.

References

[1] L. Babai and P. Frankl. *Linear Algebra Methods in Combinatorics.* University of Chicago, 1988, 1992, 2020.

[2] C. Buchanan, A. Clifton, A. Culver, J. Nie, J. O'Neill, P. Rombach, and M. Yin. Odd Covers of Graphs, *arXiv:*, 2022.

[3] C. Buchanan, C. Purcell, and P. Rombach. Subgraph complementation and minimum rank. *The Electronic Journal of Combinatorics*, 29(1), 2022.

[4] A. Mohammadian. Trees and acyclic matrices over arbitrary fields. *Linear and Multilinear Algebra*, 64(3):466–476, 2016.

[5] J. Radhakrishnan, P. Sen, and S. Vishwanathan. Depth-3 arithmetic circuits for $S_n^2(x)$ and extensions of the Graham-Pollack [*sic*] theorem. *Foundations of Software Technology and Theoretical Computer Science*, Springer, Berlin, Heidelberg, 2000.