Geometric Riemann Sums

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1 The problem

We would like to evaluate the definite integral of a continuous function f on a closed interval $[a, b]$ using Riemann sums. In the canonical example, we partition $[a, b]$ into k intervals of equal size and evaluate the sum of the areas of rectangles of width $\frac{b-a}{k}$ and height $f(x_i)$ where x_i is some point in the ith interval. We obtain a sum of the form

$$\sum_{1}^{k} f(x_i)(\tfrac{b-a}{k}),$$

which is an estimate of the area between the x-axis and the curve of $y = f(x)$ from $x = a$ to $x = b$. The more rectangles we use, the better our estimate gets, and thus we may evaluate the area under $f(x)$ by taking a limit,

$$\int_a^b f(x)dx = \lim_{k\to\infty} \sum_{i=1}^{k} f(x_i)(\tfrac{b-a}{k}).$$

Now all we need to do is evaluate this limit.... Tricky stuff. We will think about this problem a little differently using rectangles of varying widths.

2 Geometric series

First, a warm-up. You may have seen a popular proof that $1 = 0.999\ldots$. It goes like this: set

$$S = 0.999\ldots.$$

Then,

$$10S = 9.999\ldots,$$

and

$$9S = 10S - S = 9.999\ldots - 0.999\ldots = 9.$$

Dividing everything by 9, the first and last terms give us $S = 1$. But $S = 0.999\ldots$, so $1 = 0.999\ldots$.

Since $0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$, we could have phrased the proof like this: consider the system of equations

$$\begin{aligned} 10S &= 9 + \tfrac{9}{10} + \tfrac{9}{100} + \cdots \\ S &= 0 + \tfrac{9}{10} + \tfrac{9}{100} + \cdots . \end{aligned}$$

Subtracting the bottom equation from the top equation, we see that $9S = 9$ and, as before, $0.999\ldots = S = 1$.

This is the basic idea of a *geometric series*. If q is some fraction in the interval $(0, 1)$, then an infinite sum of the form

$$\sum_{n=1}^{\infty} q^n = q + q^2 + q^3 + q^4 + \cdots$$

is an example of a geometric series, and it is *always* finite (provided q stays strictly between 0 and 1). We can also multiply every term in a geometric series by a constant, as we did in the previous example. We may have expressed $0.999\ldots$ as the geometric series $\sum_1^\infty 9(\frac{1}{10})^n$.

An appropriate next question: Can we always calculate what a geometric series equals? The answer is yes. Using the same strategy as before, set $S = \sum_1^\infty q^n = q + q^2 + q^3 + \cdots$. Then $qS = q^2 + q^3 + q^4 + \cdots$, and

$$S - qS = (1 - q)S = q.$$

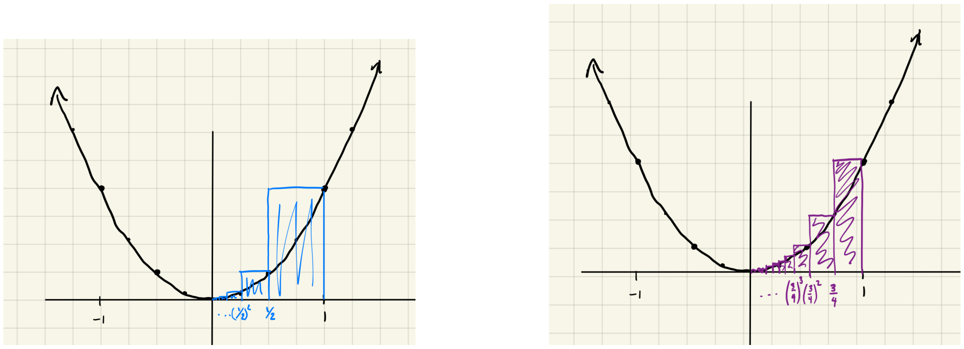
Dividing everything by $1 - q$, we obtain $S = \frac{q}{1-q}$, which is precisely the infinite sum $\sum_1^\infty q^n$. In the previous example, we obtain

$$S = 9\sum_{1}^{\infty}\left(\frac{1}{10}\right)^n = 9 \cdot \frac{1/10}{1 - (1/10)} = 1.$$

We'll remember this.

3 Geometric Riemann sums

Say we would like to evaluate the area under the curve $y = x^2$ on the interval $[0, 1]$ using Riemann sums. We can use a geometric series to make our job a



(a) $q = \frac{1}{2}$ (b) $q = \frac{3}{4}$

Figure 1: Geometric Riemann partitions of $[0, 1]$ for the curve $y = x^2$.

little easier. Instead of partitioning $[0, 1]$ into intervals of equal size, we will partition it into one interval of size $\frac{1}{2}$, one of size $\frac{1}{4}$, one of size $\frac{1}{8}$, *etc.* as follows:

$$\left[\tfrac{1}{2}, 1\right], \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \left[\tfrac{1}{8}, \tfrac{1}{4}\right], \left[\tfrac{1}{16}, \tfrac{1}{8}\right], \ldots.$$

In other words, we will take intervals of width $(\frac{1}{2})^n$ for $n = 1, 2, 3, \ldots$. If we take right-hand Riemann sums, or overestimates, our rectangles will look like the crude sketch in Subfigure 1a.

Now, this certainly isn't the best estimate of the area under x^2 between $x = 0$ and $x = 1$, but we can calculate it using a geometric series. The area of the blue rectangles, $A_{1/2}$, is the sum of their heights times their widths, or

$$\begin{aligned} A_{1/2} &= 1 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \cdot \frac{1}{8} + \cdots \\ &= \sum_0^\infty \left(\frac{1}{2^n}\right)^2 \cdot \left(\frac{1}{2}\right)^{n+1} = \sum_0^\infty \left(\frac{1}{2}\right)^{2n} \cdot \left(\frac{1}{2}\right)^{n+1} \\ &= \sum_0^\infty \left(\frac{1}{2}\right)^{3n+1}. \end{aligned}$$

Adapting the trick from the last section, if

$$A_{1/2} = \frac{1}{2} + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^{10} + \cdots,$$

then

$$\left(\frac{1}{2}\right)^3 A_{1/2} = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^{10} + \cdots,$$

and

$$A_{1/2} - \frac{1}{8}A_{1/2} = \frac{7}{8}A_{1/2} = \frac{1}{2}.$$

Multiplying every term by $\frac{8}{7}$, we see that

$$A_{1/2} = \frac{1}{2} \cdot \frac{8}{7} = \frac{4}{7}.$$

As we said, this is a pretty bad estimate of the desired area. However, if we had instead considered the intervals

$$\left[\tfrac{3}{4}, 1\right], \left[(\tfrac{3}{4})^2, \tfrac{3}{4}\right], \left[(\tfrac{3}{4})^3, (\tfrac{3}{4})^2\right], \ldots,$$

and had drawn rectangles of width $(\frac{3}{4})^n - (\frac{3}{4})^{n+1}$ for $n = 0, 1, 2, \ldots$, as in Subfigure 1b, then the infinite collection of intervals would still cover $[0, 1]$ and we could use the same process to obtain an even better estimate. In fact, we can repeat the above calculations for rectangles of width $q^n - q^{n+1}$ for *any* fraction q between 0 and 1 as follows:

$$A_q = \sum_0^\infty (q^n)^2(q^n - q^{n+1}) = \sum_0^\infty q^{3n} - q^{3n+1} = (1-q)\sum_0^\infty q^{3n}.$$

If we set $S = \sum_0^\infty q^{3n} = 1 + q^3 + q^6 + \cdots$, then $q^3S = q^3 + q^6 + \cdots$, and

$$S - q^3S = (1-q^3)S = 1.$$

Thus, $S = \frac{1}{1-q^3}$, and

$$A_q = (1-q)S = \frac{1}{1+q+q^2}.$$

As our fractions q become closer to 1, it is not too hard to see that the approximation of the area between the x-axis and the curve $y = x^2$ improves. In fact, if we can evaluate the limit of A_q as q approaches 1, we will obtain the exact area under the curve. Luckily, this is an easy limit to take, and

$$\int_0^1 x^2 dx = \lim_{q\to 1} A_q = \lim_{q\to 1} \frac{1}{1+q+q^2} = \frac{1}{3},$$

as desired.

Repeat this trick for *any* function of the form $f(x) = kx^n$ (k real and n a positive integer) for amazing results.

References

[1] George E Andrews. The geometric series in calculus. The American mathematical monthly, 105(1):36-40, 1998.

[2] Also, thanks to Blair Seidler's calculus lessons at JHU CTY.