Expressing graphs as symmetric differences of cliques

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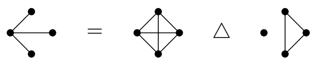
The clique-build number

Any finite simple graph $G = (V, E)$ can be represented by a collection of cliques in the complete graph on V whose symmetric difference is G. For instance, consider $\{\{u, v\} \mid uv \in E\}$. But we can often do better.

Example 1.



Question. What is the minimum cardinality of such a collection of cliques?

Definition 2. A *clique construction* of G is a collection $\mathscr{C}$ of subsets of V such that, for each pair $u, v \in V$, $uv \in E$ if and only if u and v appear together an odd number of times in $\mathscr{C}$. The minimum cardinality of a clique construction of G is the *clique-build number* of G, denoted by $c_2(G)$.

Equivalent problems

The problem of expressing a graph G as a sum of cliques modulo 2 was posed by Vatter [1].

Subgraph complementation [4]
Replace an induced subgraph of G by its graph complement.

Faithful orthogonal representations
Given a field $\mathbb{F}$, assign to each vertex of G a vector from $\mathbb{F}^d$ so that two vertices are adjacent if and only if they are represented by non-orthogonal vectors. Lovasz [3] introduced these representations over $\mathbb{R}$.

Dot product representations
Orthogonal representations in which the dot product of two vectors representing adjacent vertices is 1.

Upper bounds

A number of upper bounds for $c_2(G)$ are obtained by its equivalence to the minimum dimension of a faithful orthogonal representation of G over $\mathbb{F}_2$. Given a clique construction $\mathscr{C}$ of G, assign to each vertex v an incidence vector with a 1 in the ith slot if v appears in the ith clique in $\mathscr{C}$, and a 0 otherwise. The equivalence follows, as two vectors are orthogonal over $\mathbb{F}_2$ if and only if they share an even number of 1's.

We denote by $M(\mathscr{C})$ the *clique-incidence matrix* whose rows are the aforementioned vectors. For example, the matrix corresponding to the construction $\mathscr{C}$ in Example 1 is

$$M(\mathscr{C}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}. \qquad (1)$$

Propositions 3 and 4 are corollaries of Theorems 1 and 3 in [4], obtained by this equivalence.

Let n denote the order of a graph G.

Proposition 3. *For any graph G, $c_2(G) \leq n - 1$.*

Proposition 4. *For any graph G ($n > 2$) other than P_n, $c_2(G) \leq n - 2$.*

Theorem 5 ([2]). *For any graph G with vertex cover number $\tau(G)$, $c_2(G) \leq 2\tau(G)$.*

Proof. For each $v \in V(G)$, we can build edges to any subset $S \subseteq N(v)$ in two steps using cliques S and $S \cup \{v\}$. Given a minimum vertex cover U of G, or set of vertices spanning the edges of G, build the edges incident to each $u \in U$ which have not already been built in at most two steps. □

Notice that $c_2(G) < 2\tau(G)$ if any of the cliques we use are singletons, that is, if some vertex in U has only one neighbor outside of U.

Minimum rank

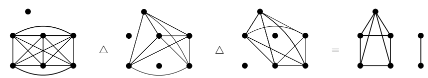
Let $M = M(\mathscr{C})$ be a clique-incidence matrix. The off-diagonal entry (i, j) of $MM^T \pmod 2$ is 0 if and only if the vertices corresponding to the ith and jth rows of M are nonadjacent.

The *minimum rank* of G over $\mathbb{F}$ is the minimum rank over all matrices in $\mathbb{F}^{n \times n}$ whose off-diagonal zeros match those of the adjacency matrix of G. Since $\operatorname{rank}(MM^T) \leq \operatorname{rank}(M) \leq c_2(G)$, we have

$$\operatorname{mr}(G, \mathbb{F}_2) \leq c_2(G). \qquad (2)$$

Theorem 6. *For any forest F and field $\mathbb{F}$, we have* $c_2(F) = \operatorname{mr}(F, \mathbb{F})$.

There is a close relationship between $c_2(G)$ and $\operatorname{mr}(G, \mathbb{F}_2)$: the numbers differ by at most 1, and do so only if $c_2(G)$ is odd. On the other hand, these invariants have important differences. The minimum rank of G with components $G_1, \ldots, G_\ell$ is $\sum_1^\ell \operatorname{mr}(G_i, \mathbb{F})$, but this is not the case for $c_2(G)$. While $c_2(W_5) = 3$ and $c_2(K_2) = 1$, we have



Theorem 7. *For any graph G, the following are equivalent.*

i. $c_2(G) = \operatorname{mr}(G, \mathbb{F}_2) + 1$;

ii. there is a unique matrix A of minimum rank over $\mathbb{F}_2$ which fits G, and every diagonal entry of A is 0;

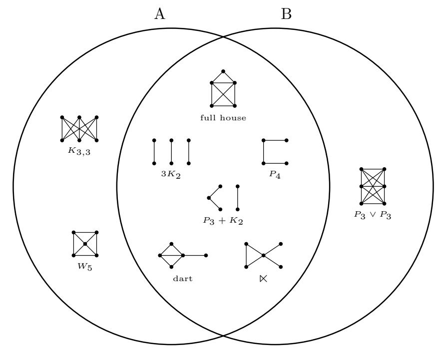
iii. there is an optimal clique construction of G in which every vertex appears an even number of times;

iv. for every component G' of G, $c_2(G') = \operatorname{mr}(G') + 1$.

Forbidden subgraphs

The graph property $c_2(G) \leq k$ is hereditary. We have shown in [2] that it is defined by a finite set of minimal forbidden induced subgraphs. For odd k, we have $c_2(G) = k$ whenever $\operatorname{mr}(G, \mathbb{F}_2) = k$, and $c_2(G) \leq k$ whenever $\operatorname{mr}(G, \mathbb{F}_2) < k$. Thus, the sets of minimal forbidden induced subgraphs for $\{G : \operatorname{mr}(G, \mathbb{F}_2) \leq k\}$ and $\{G : c_2(G) \leq k\}$ are the same.

This is not the case when k is even. We exhibit the sets of minimal forbidden induced subgraphs for $c_2(G) \leq 2$ and $\operatorname{mr}(G, \mathbb{F}_2) \leq 2$, labeled A and B respectively, below.



References

[1] V. Vatter. Terminology for expressing a graph as a sum of cliques (mod 2), URL (version: 2018-12-15): https://mathoverflow.net/q/317716

[2] C. Buchanan, C. Purcell, and P. Rombach. Expressing graphs as symmetric differences of cliques of the complete graph, *arXiv:2101.06180*, 2021.

[3] L. Lovász. On the Shannon capacity of a graph. IEEE Transactions on Information Theory, 1979.

[4] V. Alekseev and V. Lozin. On orthogonal representations of graphs. Discrete Mathematics, 2001.