

On the last new vertex visited by a random walk in a directed graph

Calum Buchanan

University of Vermont

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Joint work with Paul Horn and Puck Rombach

Cover tours

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Cycles and complete graphs have the property that a random cover tour, starting at any vertex, is equally likely to end at any other vertex.

Ronald Graham asked if there are any other such graphs.

Undirected graphs

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Lemma (Lovász-Winkler, 1993)

If G is connected and $uv \notin E(G)$, then there is a neighbor x of u such that $\mathbb{P}(L(x, v)) \leq \mathbb{P}(L(u, v))$. Further, this inequality is strict if there is a cover tour of G from u to v which does not revisit u .

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Theorem (Lovász-Winkler, 1993)

Cycles and complete graphs are the only undirected graphs with the property that $\mathbb{P}(L(u, v)) = \mathbb{P}(L(u, w))$ for any three distinct vertices u, v , and w .

Directed graphs

We denote by $L(u, v)$ the event that v is the last vertex visited by a random cover tour of a **digraph** G starting at vertex u .

Lemma (B.-Horn-Rombach, 2023)

If G is **strongly connected** and $uv \notin E(G)$, then there is an **out-neighbor** x of u such that $\mathbb{P}(L(x, v)) \leq \mathbb{P}(L(u, v))$.

Further, this inequality is strict if there is a cover tour from u to v which does not revisit u .

Theorem (B.-Horn-Rombach, 2023)

Cycles and complete graphs¹ are the only **directed** graphs with the property that $\mathbb{P}(L(u, v)) = \mathbb{P}(L(u, w))$ for any three distinct vertices u , v , and w .

¹with all edges considered bidirected

Proof of theorem

It suffices to show that, in any digraph with the property that $\mathbb{P}(L(u, v)) = \mathbb{P}(L(u, w))$ for any three distinct vertices u , v , and w , every edge is bidirected.

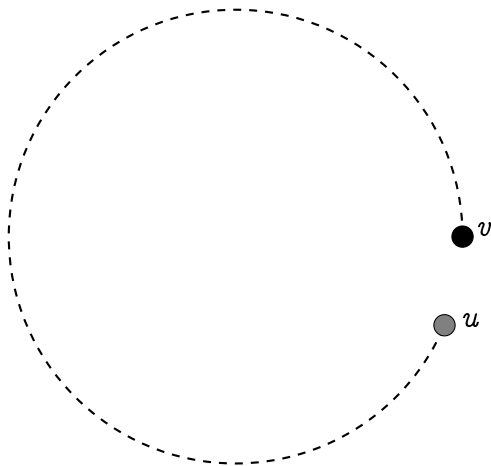
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By our lemma, if G is a digraph with the above property, and if T is a cover tour of G from u to v , then either $uv \in E(G)$ or u appears at least twice in T .

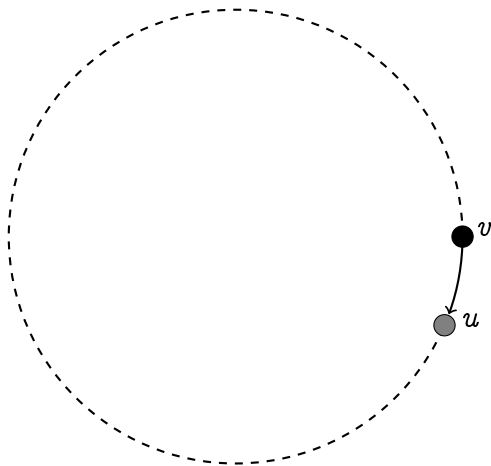
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Suppose, for a contradiction, that $uv \notin E(G)$ but $vu \in E(G)$.
Consider a cover tour T from u to v of minimum length.



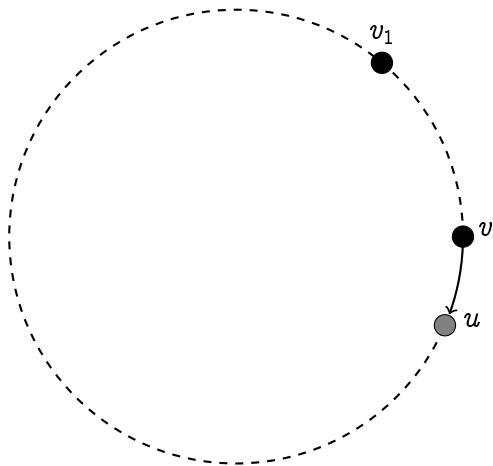
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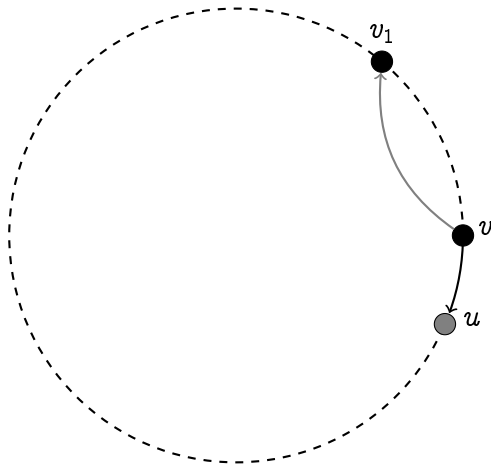
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Let v_1 be the last new vertex visited by the walk which first takes vu , then follows T .



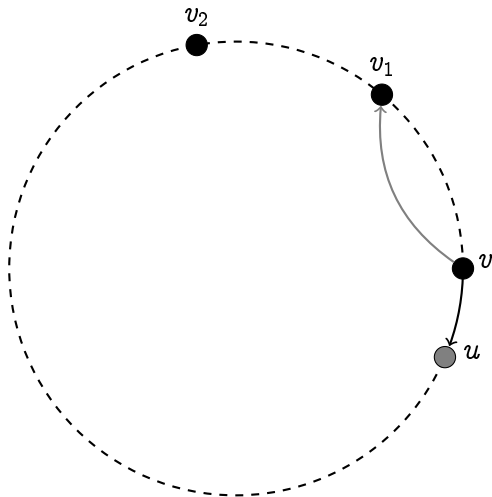
Proof of theorem

Let v_1 be the last new vertex visited by the walk which first takes vu , then follows T . Since v appears only once, $vv_1 \in E(G)$.



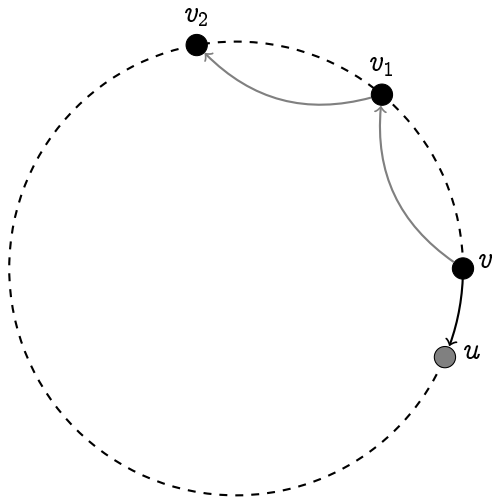
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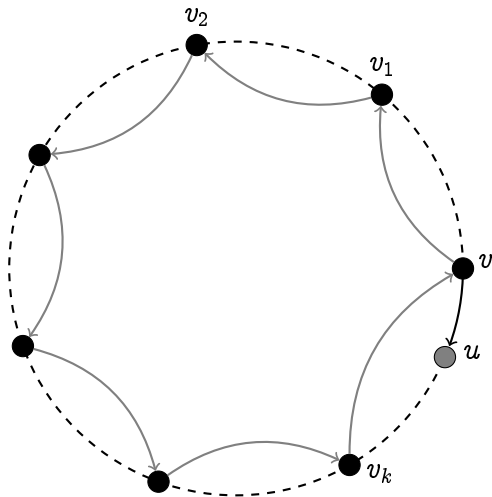
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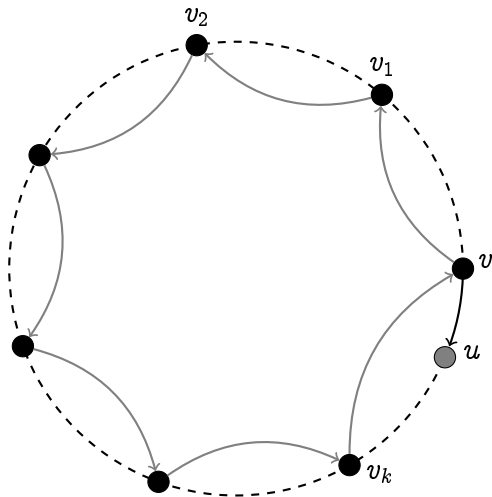
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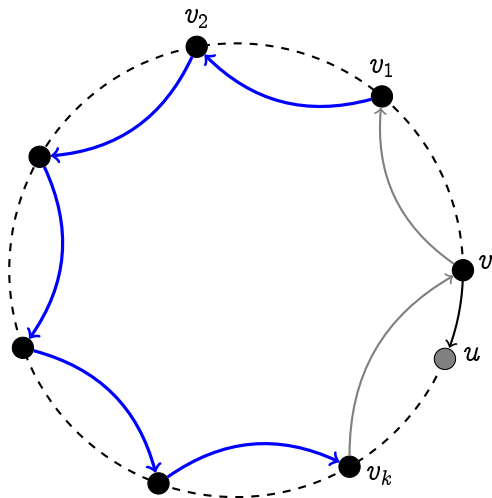
Proof of theorem

We now show that $v_1 v \in E(G)$ by finding a cover tour from v_1 to v which visits v_1 only once.



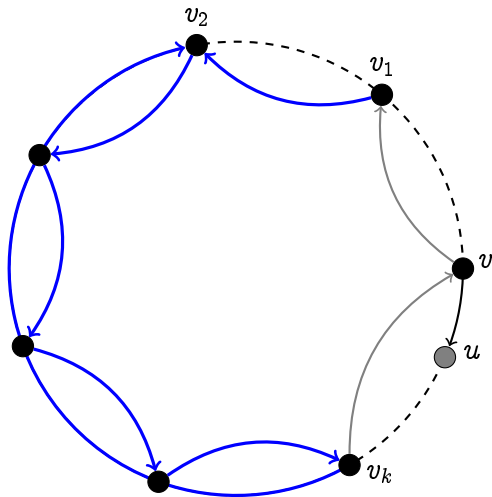
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We now show that $v_1 v \in E(G)$ by finding a cover tour from v_1 to v which visits v_1 only once. Part (I) starts at v_1



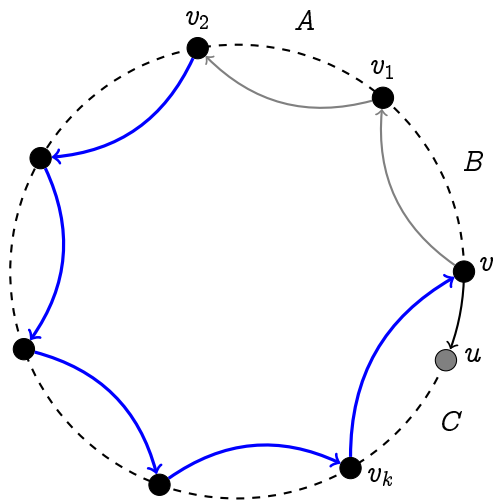
Proof of theorem

We now show that $v_1 v \in E(G)$ by finding a cover tour from v_1 to v which visits v_1 only once. Part (I) starts at v_1 and stops at v_2 .



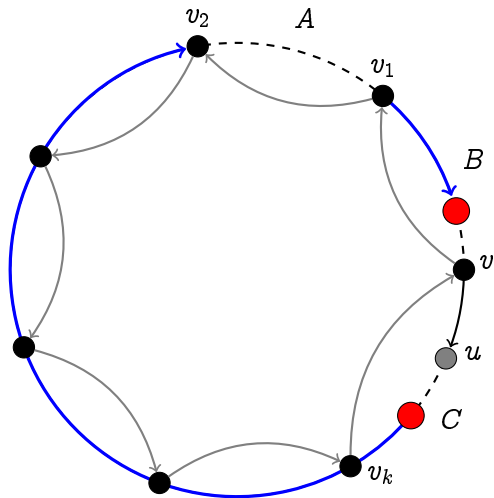
Proof of theorem

Part (II) starts at v_2 and ends the cover tour at v .



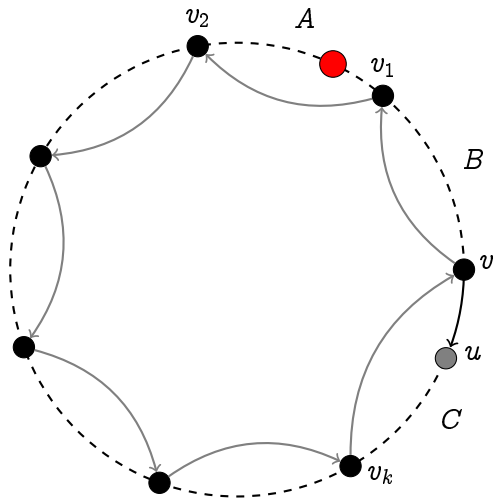
Proof of theorem

If there is an unseen vertex in B and C , Part (I) becomes:



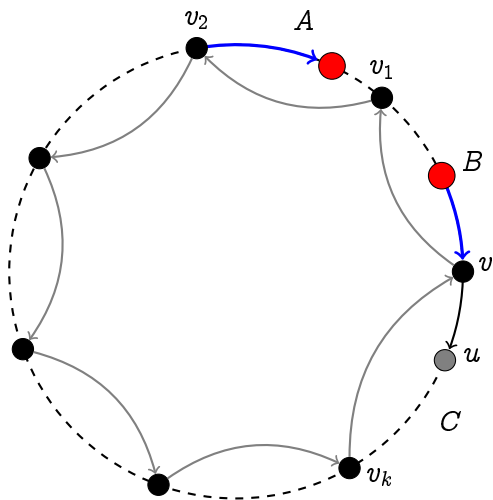
Proof of theorem

Any remaining unseen vertices have one copy in A and another in B or C . Let y be the last unseen vertex in A .



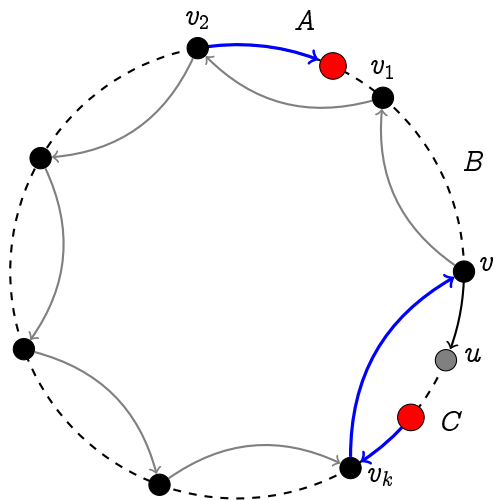
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If y has a copy in B , Part (II) becomes:



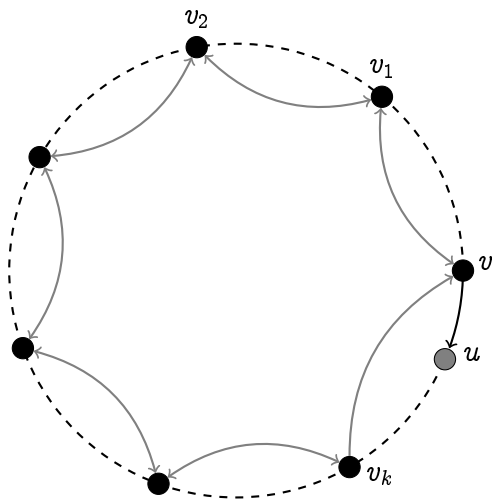
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Otherwise, y has a copy in C . Part (II) becomes:



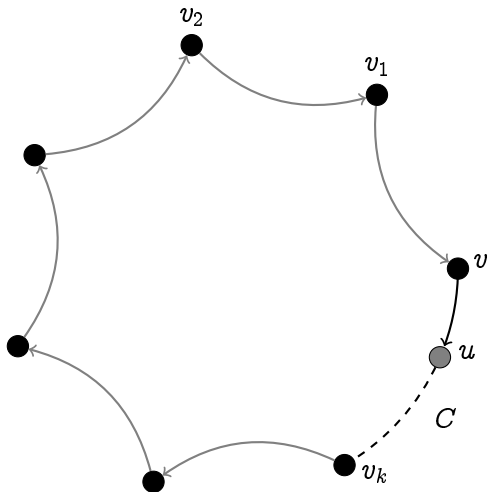
Proof of theorem

Similarly, we have $v_{i+1}v_i \in E(G)$ for each $i \in \{1, \dots, k\}$.





Proof of theorem

In fact, the edge v_1v and each $v_{i+1}v_i$ is in T , by minimality.
But u appears twice in T , and not twice in C , a contradiction.



Thank you!

References

-  C. Buchanan, P. Horn, P. Rombach, On the last new vertex visited by a random walk in a directed graph, *Discrete Math. Lett.*, Vol. 11, (2023) 96-98.
-  L. Lovász, P. Winkler, A note on the last new vertex visited by a random walk, *J. Graph Theory*, Vol. 17, No. 5, (1993) 593-596.