REAL ANALYSIS PHD QUALIFYING EXAM
September 20, 2007

A passing grade is 6 problems done completely correctly, or 5 done completely correctly with substantial progress on 2 others.

1. Let \((X, d)\) be a compact metric space, where we take “compact” to mean “every open cover of \(X\) has a finite subcover.” Show that every sequence \(\{x_n\}_{n=1}^\infty\) in \(X\) has a subsequence converging to some \(z \in X\).

2a). Let \(\{b_n\}\) be a sequence of positive numbers which is unbounded: \(\sup_n b_n = \infty\). Show that there is a sequence of positive numbers \(\{a_n\}\) such that \(\sum a_n < \infty\), but for which \(\sum a_n b_n = \infty\).

2b) Let \(\{c_n\}\) be a sequence of positive numbers such that \(\sum c_n = \infty\). Show that there is a sequence of positive numbers \(\{d_n\}\) such that \(\lim_n d_n = 0\), but for which \(\sum c_n d_n = \infty\).

3. Prove the following: If \(f\) is differentiable on \((0,1)\) and \(f'(1/4) < 0 < f'(3/4)\), there is a \(c \in (1/4, 3/4)\) such that \(f'(c) = 0\).

4. Let \(\{f_n\}\) be a sequence of functions in \(L^p(\mathbb{R}, \mathcal{L}, m)\), where \(1 < p < \infty\), \(\mathcal{L}\) is the Lebesgue measurable sets, and \(m\) denotes Lebesgue measure. Suppose that

\[
\sup_n \|f_n\|_p < \infty. \tag{1}
\]

Show that \(\{f_n\}\) is uniformly integrable, which means: for every \(\epsilon > 0\) there is a \(\delta > 0\) such that, for all \(E \in \mathcal{L}\), \(m(E) < \delta\) implies

\[
\sup_n \int_E |f_n| \, dm < \epsilon.
\]

Also, give an example of a sequence in \(L^1\) satisfying (1) for \(p = 1\), but which is not uniformly integrable.

5. Let \(f\) and \(g\) belong to \(L^2(\mathbb{R}, \mathcal{L}, m)\). Show that

\[
\lim_{n \to \infty} \int f(x) g(x + n) \, dm = 0.
\]

6. Let \(\| \cdot \|\) denote the usual Euclidean norm in \(\mathbb{R}^d\). Let \(A\) and \(B\) be non-empty subsets of \(\mathbb{R}^d\), where \(A\) is compact and \(B\) is closed (with respect to the usual topology). Show that there exist points \(a \in A\) and \(b \in B\) such that, for all \(x \in A\) and \(y \in B\),

\[
\|a - b\| \leq \|x - y\|.
\]

Show that such points need not exist if \(A\) is merely assumed to be closed.
7. Let \((X, \mathcal{M}, \mu)\) be a measure space, and suppose that \(\{E_n\}_1^\infty\) is a sequence from \(\mathcal{M}\) with the property that 
\[
\lim_{n \to \infty} \mu(X \setminus E_n) = 0, 
\]
Let \(G \subset X\) be the set of \(x\)'s that belong to only finitely many of the sets \(E_n\); i.e., \(x \in G\) if and only if \(x\) belongs to at most finitely many \(E_n\)'s. Show that \(G \in \mathcal{M}\) and \(\mu(G) = 0\).

8. Show that, if \(f \in L^1(\mathbb{R}, \mathcal{L}, m)\),
\[
\lim_{n \to \infty} \int f(x) (\sin(nx))^2 \, dm = (1/2) \int f(x) \, dm.
\]

9. Suppose that \(\{f_n\}\) is a sequence in \(L^2(\mathbb{R}, \mathcal{L}, m)\) such that \(\sum_1^\infty \|f_n\|_2 < \infty\) and \(\sum_1^\infty f_n(x) = 0\) for (Lebesgue-)almost-every \(x \in \mathbb{R}\). Prove that, for all \(g \in L^2(\mathbb{R}, \mathcal{L}, m)\),
\[
\sum_1^\infty \int f_n(x) g(x) \, dm
\]
exists and equals 0.

10. Define \(f : \mathbb{R}^2 \mapsto \mathbb{R}^2\) by \(f(x, y) = (x^2 - y, x^4 + y^2)\), and let \((a, b) \in \{(x, y) : x < 0, y > 0\}\). Show that \(f\) is one-to-one on some open set \(U\) containing \((a, b)\), and that there is a differentiable \(g : f[U] \mapsto U\) such that \(f(g(x, y)) = (x, y)\) for all \((x, y) \in U\). In other words, prove that, at every point in the open second quadrant, \(f\) has a locally defined differentiable inverse.