REAL ANALYSIS PH.D. QUALIFYING EXAM SOLUTION SET

January 31, 2009

A passing paper consists of 7 problems solved completely, or 6 solved completely with substantial progress on 2 others.

1. Let (X, d) be a metric space. A set $E \subseteq X$ is called *discrete* if there is $\delta > 0$ such that, for all x and y in E with $x \neq y$ we have $d(x, y) > \delta$. Show that a discrete set is necessarily closed. (Use any standard definition of "closed set" in a metric space.)

Solution. Let $\{x_n\} \subset E$ be a sequence and $x_n \to p$. The sequence $\{x_n\}$ is Cauchy; E's discreteness forces it to be eventually constant. Therefore $x_n = p$ for all sufficiently large n, and $p \in E$.

2. Suppose that $f : (0, 1) \to \mathbb{R}$ is differentiable on all of $(0, 1)$ and $f'(1/4) < 0 < f'(3/4)$. Show that there is a $c \in (1/4, 3/4)$ such that $f'(c) = 0$.

Solution. Since f is continuous on $[1/4, 3/4]$, it attains a minimum at some $c \in [1/4, 3/4]$. Having $f'(1/4) < 0$ implies that $f(x) < f(1/4)$ for some $x > 1/4$ and close to $1/4$; therefore $c \neq 1/4$. Having $f'(3/4) > 0$ implies that $f(x) < f(3/4)$ for some $x < 3/4$ and close to $3/4$; therefore $c \neq 3/4$. Therefore $c \in (1/4, 3/4)$ and, since c is a minimum, $f'(c) = 0$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on all of $\mathbb{R}$ and $\lim_{x\to\infty} f'(x) = A$, where A is a real number. Show that $\lim_{x\to\infty} \dfrac{f(x)}{x}$ exists and equals A. [Hint: Show this for $A = 0$ first.]

Solution. Following the hint, first treat the case of $A = 0$. Let $\epsilon > 0$, and let $a > 0$ be so large that $t > a$ implies $|f'(t)| < \epsilon$. If $x > a$ then, by the Mean Value Theorem, there is a point $a < c_x < x$ such that

$$f(x) = f(a) + f'(c_x)(x - a);$$

therefore

$$\frac{f(x)}{x} = \frac{f(a)}{x} + f'(c_x)(1 - (a/x)) \equiv (I) + (II).$$

Term $(I) \to 0$ as $x \to \infty$ and $|(II)| < \epsilon$. Therefore $|f(x)/x| < 2\epsilon$ when x is large enough. That proves the $A = 0$ case. For general A, set $g(x) = f(x) - Ax$. Then $g'(x) \to 0$, implying

$$g(x)/x = \frac{f(x)}{x} - A \to 0$$

as $x \to \infty$.

4. Let $f : [1, \infty) \to [0, \infty)$ be a non-increasing function. Prove that

$$\int_1^\infty f(x)\,dx < \infty \qquad \text{if and only if} \qquad \sum_{k=0}^{\infty} 2^k f(2^k) < \infty.$$

Solution. For every $k = 0, 1, 2, \ldots,$

$$f(2^k)(2^{k+1} - 2^k) \geq \int_{2^k}^{2^{k+1}} f(x)\,dx \geq f(2^{k+1})(2^{k+1} - 2^k).$$

But $2^{k+1} - 2^k = 2^k$. Therefore

$$\sum_{k=0}^{\infty} 2^k f(2^k) \geq \int_1^{\infty} f(x)\, dx \geq \sum_{k=0}^{\infty} 2^k f(2^{k+1}).$$

But the sum on the far right equals

$$(1/2)\left(\sum_{k=1}^{\infty} 2^k f(2^k)\right).$$

5. Consider the two surfaces in $\mathbb{R}^3$,

$$\begin{aligned}\Sigma_1 &= \{(x,y,z) :\ z = xy\}\\ \Sigma_2 &= \{(x,y,z) :\ x^2+y^2+z^2 = 1\},\end{aligned}$$

and let $\Gamma \equiv \Sigma_1 \cap \Sigma_2$. For almost all of the points $(\tilde{x}, \tilde{y}, \tilde{z}) \in \Gamma$, the Implicit Function Theorem guarantees the existence of a differentiable function $g = (g_1, g_2)$, defined on some open neighborhood U of $\tilde{z}$ and mapping into $\mathbb{R}^2$, such that $(g_1(z), g_2(z), z) \in \Gamma$ for all $z \in U$. But there are FOUR points $(\tilde{x}, \tilde{y}, \tilde{z})$ where the IFT does not guarantee the existence of such a g. Find the points, with justification.

Solution. The curve Γ is the simultaneous zero set of $\phi_1(x,y,z) = xy - z$ and $\phi_2(x,y,z) = x^2+y^2+z^2-1$. The appropriate Jacobian matrix has determinant $2(y^2 - x^2)$. The "bad" points are where $|x| = |y| \equiv \alpha$. Setting $|z| = |x||y| = \alpha^2$, and plugging this into $x^2+y^2+z^2-1 = 0$, we get $\alpha^4 + 2\alpha^2 - 1 = 0$, implying $\alpha^2 = -1 + \sqrt{2}$ (we take only the positive root from the quadratic formula). The points are given by $x = \pm\sqrt{-1+\sqrt{2}}$, $y = \pm\sqrt{-1+\sqrt{2}}$ (independent plus or minus), and $z = xy$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space, where $\mathcal{M}$ is a σ-algebra, and let $g : X \to [0,\infty]$ be a non-negative measurable function. For each $E \in \mathcal{M}$ define

$$\nu(E) = \int g\, \chi_E\, d\mu.$$

(a) Show that $\nu(E)$ defines a measure on $\mathcal{M}$. (You may quote without proof any standard theorems from measure theory in your argument.)

(b) In a similar fashion, show that if $f : X \to [0,\infty]$ is any non-negative measurable function, then $\int f\, d\nu = \int f\, g\, d\mu$.

Solution. a) The only nontrivial part is countable additivity. Let $\{E_k\} \subset \mathcal{M}$ be disjoint. Then:

$$\begin{aligned}
\nu(\cup E_k) &= \int g\, \chi_{\cup E_k}\, d\mu \\
&= \int g \left(\sum_k \chi_{E_k} \right) d\mu \\
&= \lim_{n\to\infty} \int g \left(\sum_1^n \chi_{E_k} \right) d\mu \qquad (1) \\
&= \lim_{n\to\infty} \sum_1^n \nu(E_k) \\
&= \sum_1^\infty \nu(E_k),
\end{aligned}$$

where (1) follows from the Monotone Convergence Theorem.

b) If $f = \chi_E$ the result follows from the definition of ν. By linearity it is true for any non-negative simple function. If f is any non-negative measurable function, there is a sequence of non-negative simple functions $0 \le \phi_1 \le \phi_2 \le \phi_3 \le \cdots$ such that $\phi_n \to f$ pointwise. By Monotone Convergence,

$$\int f\, d\nu = \lim_{n\to\infty} \int \phi_n\, d\nu = \lim_{n\to\infty} \int \phi_n\, g\, d\mu = \int f\, g\, d\mu.$$

7. Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space, and $\{E_k\}$ is a sequence of sets from $\mathcal{M}$ such that $\mu(E_k) > 1/100$ for all k. Let F be the set of points $x \in X$ which belong to infinitely many of the sets E_k.

- **(a)** Show that $F \in \mathcal{M}$, i.e., F is a measurable set.
- **(b)** Prove that $\mu(F) \ge 1/100$.
- **(c)** Give an example to show that conclusion (b) can fail if $\mu(X) = \infty$.

Solution. a) A point x belongs to F if and only if, for all j, there is a $k \ge j$ such that $x \in E_k$. Therefore $F = \cap_j \cup_{k\ge j} E_k$, which belongs to $\mathcal{M}$. b) Put $D_j = \cup_{k\ge j} E_k$. The sets D_j are decreasing, their intersection is F, and $\mu(D_1) \le \mu(X) < \infty$. Therefore $\mu(F) = \lim_{j\to\infty} D_j$. But, for all j, $\mu(D_j) \ge \mu(E_j) > 1/100$. c) With $\mu =$ Lebesgue measure on $\mathbb{R}$, let E_j be the interval $(j, j+1)$.

8. Find the value of

$$\lim_{n\to\infty} \int_0^\infty \left(\sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x}\, dx,$$

and justify your assertion by quoting appropriate facts from calculus and one or more limit theorems from measure theory.

Solution. The absolute values of all of the partial sums of the power series are pointwise bounded by e^x (look at *its* power series); therefore the integrands are all bounded by $e^x e^{-2x} = e^{-x}$, which is integrable on $[0, \infty)$. The power series conveges pointwise to $\cos x$. Therefore, by the Dominated Convergence Theorem, the limit equals

$$\int_0^\infty \cos x\, e^{-2x}\, dx.$$

We find the integral by doing 2 integrations by parts; its value is 2/5.

9. Let $(X, ||\quad||)$ be a normed linear space.

(a) State what it means for $(X, ||\quad||)$ to be a Banach space, and give an example, with details, of a normed linear space that is **not** a Banach space.

(b) Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X and let

$$S_N = \sum_{k=1}^{N} x_k$$

be the usual N^{th} partial sum of the series $\sum_{k=1}^{\infty} x_k$. The series is said to be *summable* if the sequence $\{S_N\}_{N=1}^{\infty}$ of partial sums converges to an element of X. The series is called *absolutely summable* if $\sum_{k=1}^{\infty} ||\, x_k \,|| < \infty$.

Prove that $(X, ||\quad||)$ is a Banach space if and only if every absolutely summable series is summable. (You may use without proof the fact that if a Cauchy sequence has a subsequence that converges to L, then the entire sequence also converges to L.)

Solution. a) A Banach space is a NLS that is complete with respect to the norm-induced metric. For the example (one possibility), first let $Y = \ell^{\infty}$, the bounded sequences on $\mathbb{N}$ with the supremunm norm. Let $X \subset Y$ be the *finite* (eventually 0) sequences on $\mathbb{N}$, with the same norm. Consider the sequence $\{x_n\}$ from X defined by:

$$x_n = \{1,\ 1/2,\ 1/3,\ 1/4,\ \ldots,\ 1/n,\ 0,\ 0,\ 0,\ \ldots\}.$$

The sequence $\{x_n\}$ is Cauchy (with respect to the norm) because it has a limit p in Y. But that p does not belong to X. Therefore $\{x_n\}$ can have no limit in X, because limits are unique.

b) Suppose $(X, ||\quad||)$ is a Banach space, and $\sum_{k=1}^{\infty} ||\, x_k \,|| < \infty$. Let N be so large that $\sum_{k>N} ||\, x_k \,|| < \epsilon$. Recall that $S_n = \sum_1^n x_k$. If $N \le m < n$ then $\|S_m - S_n\| \le \sum_{m+1}^{n} \|x_k\| < \epsilon$. Therefore $\{S_n\}$ is Cauchy and the series is summable. Conversely, suppose that every absolutely summable series is summable, and that $\{x_n\} \subset X$ is Cauchy. Let n_0 be the least N such that if m and n are $\ge N$ then $\|x_n - x_m\| < 1$. Then, having chosen $n_0 < n_1 < n_2 < \cdots < n_p$, let n_{p+1} be the least $N > n_p$ such that if if m and n are $\ge N$ then $\|x_n - x_m\| < 2^{-p-1}$. The series

$$x_{n_0} + (x_{n_1} - x_{n_0}) + (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \cdots \equiv \eta_0 + \eta_1 + \eta_2 + \cdots$$

is absolutely summable because $\|\eta_k\| \le 2^{-k}$ for $k > 0$. Therefore it is summable. Call its sum y. For any $p \ge 0$ (telescoping series),

$$\sum_{k=0}^{p} \eta_k = x_{n_p}.$$

Therefore $x_{n_p} \to y$ as $p \to \infty$ and, because $\{x_n\}$ is Cauchy, the entire sequence converges to y.

10. Let $\phi \in L^{\infty}(\mathbb{R})$ (the measure on $\mathbb{R}$ is the usual Lebesgue measure). Show that

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx \right)^{1/n}$$

exists and equals $\|\phi\|_\infty$.

Solution. If $\|\phi\|_\infty = 0$ the problem is trivial; therefore we assume that $\|\phi\|_\infty > 0$. For all n,

$$\left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx\right)^{1/n} \leq \|\phi\|_\infty \pi^{1/n} \to \|\phi\|_\infty \tag{2}$$

as $n \to \infty$. On the other hand, for all $0 < \epsilon < \|\phi\|_\infty$ there is a Lebesgue measurable set $E \subset \mathbb{R}$ with positive measure such that $|\phi(x)| > \|\phi\|_\infty - \epsilon$ everwhere on E. Therefore, for all n,

$$\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx \geq (\|\phi\|_\infty - \epsilon)^n \int_E (1+x^2)^{-1}\, dx.$$

But, because $(1+x^2)^{-1} > 0$ everywhere, the integral

$$\int_E (1+x^2)^{-1}\, dx$$

is positive. Call its value δ. Therefore, for all n,

$$\left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx\right)^{1/n} \geq (\|\phi\|_\infty - \epsilon)\delta^{1/n} \to \|\phi\|_\infty - \epsilon. \tag{3}$$

Inequality (2) implies that

$$\limsup_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx\right)^{1/n} \leq \|\phi\|_\infty$$

and inequality (3) implies that, for all $\epsilon > 0$,

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx\right)^{1/n} \geq \|\phi\|_\infty - \epsilon,$$

which implies that the limit inferior is $\geq \|\phi\|_\infty$.