The University of Vermont

METHODS OF APPLIED MATHEMATICS

Comprehensive Examination

January 2010

NAME:
1. Show that the solution to the nonhomogenous Fredholm equation

\[ y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \]

where \( K(x,t) \) is a Hilbert-Schmidt kernel, \( f(x) \) and \( y(x) \) are square integrable functions, and \( \lambda \) is not an eigenvalue of the homogeneous Fredholm equation, has solution of the form

\[ y(x) = f(x) - \lambda \int_a^b \Gamma(x,t;\lambda)f(t)dt. \]

Here, the resolvent kernel \( \Gamma(x,t;\lambda) \) can be expressed in terms of eigenvalues \( \lambda_k \) and orthonormal eigenfunctions \( \phi_k(x) \) of the homogeneous Fredholm equation as

\[ \Gamma(x,t;\lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k(t)}{\lambda - \lambda_k}, \quad (a \leq x \leq b, a \leq t \leq b) \]

2a. Verify the following order relations:

(i) \( \varepsilon^2 \ln \varepsilon = o(\varepsilon) \) as \( \varepsilon \to 0^+ \).
(ii) \( \sin \varepsilon = O(\varepsilon) \) as \( \varepsilon \to 0^+ \).

2b. Given that \( f(x) \) is continuous and has the asymptotic representation \( f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n} \) as \( x \to \infty \), show that

\[ F(x) = \int_x^{\infty} \left( f(t) - a_0 - \frac{a_1}{t} \right) dt \sim \sum_{n=1}^{\infty} \frac{a_{n+1}}{n x^n}. \]

3. Obtain the leading-order solutions of period \( 2\pi \) of the equation

\[ \ddot{x} + \Omega^2 x - \varepsilon x^2 = \Gamma \cos t, \quad \varepsilon > 0 \]

when

(A) \( \Omega \) is far from resonance and not close to an integer;
(B) \( \Omega \approx 1 \) and \( \Gamma \) is small. Note: In (B), assume that \( \Omega^2 = 1 + \varepsilon \beta \) and \( \Gamma = \varepsilon \gamma \).

4. Find the WKB approximation to the equation

\[ \varepsilon^2 y'' - (1 + x)^2 y = 0, \quad x > 0 \]

with boundary conditions \( y(0) = 0 \) and \( \lim_{x \to \infty} y(x) = 0 \).
5. Use the method of steepest descent to obtain the asymptotic expansion for the integral

\[ f(x) = \int_0^3 \ln t \ e^{ixt} dt, \quad x \to \infty. \]

6. Construct a leading-order approximation to the solution, which is uniformly valid on \( 0 \leq x \leq 1 \) for the problem

\[ \varepsilon y'' + 2y' + y = 0, \quad y(0) = 0, \quad y(1) = 1 \]

where \( \varepsilon \) is small and positive.

7. The modified Bessel function \( I_n(x) \) has the integral representation

\[ I_n(x) = \frac{1}{\pi} \int_0^\pi \exp(x \cos \theta) \cos n\theta \, d\theta. \]

Show that \( I_n(x) \sim \frac{e^x}{(2\pi x)^{1/2}}, \quad x \to \infty. \)

8. Use the Fredholm Alternative Theorem (without proof) to find conditions under which the nonhomogenous integral equation has a solution

\[ y(x) = f(x) + \lambda \int_0^\pi \left( \cos^2 x \cos 2t + \cos 3x \cos^3 t \right) y(t) \, dt \]

for the following cases; \( A \) \( f(x) = \cos x \) and \( B \) \( f(x) = x \). Here \( \lambda \) is an eigenvalue of the corresponding homogeneous integral equation. (Hint: the values of \( \lambda \) should be obtained first.)

9. Find the asymptotic representation for \( f(z) \), \( |z| \to \infty \) in the sector \( 0 < |\arg z| < \pi/2 \), where

\[ f(z) = \int_0^\infty \frac{e^{-zt}}{1 + t^4} \, dt. \]