DIFFERENTIAL EQUATIONS

Qualifying Examination **January 10, 2008**

INSTRUCTIONS: *Two problems from each Section must be completed, and one additional problem from each Section must be attempted. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations, but do not need to finish them.* **Numeric criteria for passing***: A problem is considered completed (attempted) if a grade for it is* $\geq 85\%$ *(*$\geq 60\%$*).*

Time allowed: 3 hours.

Section 1

Problem 1

(a) Convert the following 2nd-order IVP into a 1st-order system and solve explicitly:

$$\ddot{x} - 9\dot{x} - 10x = 0 \tag{1}$$
$$x(0) = 1 \tag{2}$$
$$\dot{x}(0) = 0 \tag{3}$$

(b) Draw the phase portrait associated with (1) and determine the stability of the fixed point at the origin. Be sure to preserve proportions in his phase portrait.

Problem 2

The ω-limit set of a trajectory $\Gamma(t)$ is the set of points p such that there exists a sequence $t_n \to \infty$ with

$$\lim_{n\to\infty} \Gamma(t_n) = p \tag{1}$$

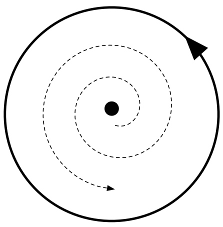

Figure 1: Example phase portrait of a 2-D system of ODE's having a trajectory $\Gamma(t)$ (dashed curve) with an ω-limit set consisting of a single limit cycle (solid curve). The fixed point in the figure is unstable (repelling).

Assuming all orbits are bounded, sketch the phase portrait of a 2-D system of ODE's having a trajectory $\Gamma(t)$ with an ω-limit set consisting of:

(a) a single limit cycle and a single fixed point;

(b) two limit cycles and a single fixed point;

(c) two limit cycles and two fixed points.

Hint: A limit cycle may join two fixed points (heteroclinic), join a fixed point to itself (homoclinic), or contain no fixed points (as in Figure 1 above).

Problem 3

Solve the following inhomogeneous IVP explicitly:

$$\dot{\mathbf{X}} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t \\ 1 \end{pmatrix} \qquad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

Problem 4

Given the 1-D ODE

$$\dot{x} = \frac{\sin(x)}{x} - \mu \,, \tag{1}$$

(a) Classify the stability of all fixed points for $\mu = 0$;

(b) Draw a bifurcation diagram for (1);

(c) Find *all* values of μ for which there exactly 2 fixed points.

Section 2

Problem 5

(a) Prove the Convolution theorem. Namely, let $f(s)$ and $g(s)$ be square integrable on the infinite line, so that their Fourier transforms $F[f](\omega)$ and $F[g](\omega)$ exist, where

$$F[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega s} f(s) ds \,,$$

and similarly for $F[g]$. **Show that**

$$F[f * g] = F[f] \cdot F[g] \,,$$

where

$$f * g\,(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s - s_1) g(s_1) ds_1 \,.$$

(b) Consider the Heat equation on the infinite line:

$$u_t = u_{xx}, \qquad u(x, 0) = q(x) \,,$$

where $q \to 0$ and $q_x \to 0$ sufficiently fast as $|x| \to \infty$. **Show that**

$$u(x, t) = \int_{-\infty}^{\infty} \phi(x - x_1, t) q(x_1) dx_1 \,,$$

where

$$\phi(x, t) = \frac{1}{\sqrt{4\pi t}} \, e^{-x^2/(4t)} \,.$$

Hint 1: Use the result of part (a).
Hint 2: You will need the value of the following integral:

$$\int_{-\infty}^{\infty} e^{-\omega^2 a + i\omega b} d\omega = \sqrt{\frac{\pi}{a}} \, e^{-b^2/(4a)} \,, \qquad a > 0 \,.$$

Problem 6

Consider the boundary value problem (BVP)

$$\begin{aligned} &u_{xx} + u = a, \quad a = \text{const}, \\ &u(0) = 0, \quad u(\pi) = 1 \,. \end{aligned} \tag{I}$$

(a) Find a (simple) change of variables from u to a new variable v that reduces (I) to a BVP with homogeneous boundary conditions:

$$\begin{aligned} &v_{xx} + v = f(x), \\ &v(0) = 0, \quad v(\pi) = 0 \,. \end{aligned} \tag{II}$$

Also, obtain the explicit form of $f(x)$.
(b) Find a formal series solution of (II).
(c) What value(s) should the constant a in (I) have in order for the solution obtained in part (b) to exist?

Problem 7

Find the displacement $u(r, \theta, t)$ of a semi-circular membrane (see the figure on the left) which satisfies the following BVP:

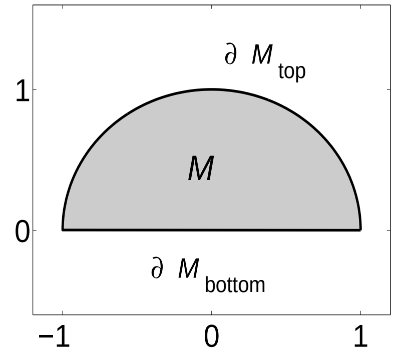

Figure 2

$$u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \qquad r, \theta \in M \qquad (1)$$

$$u_\theta = 0, \qquad r, \theta \in \partial M_{\text{bottom}} \qquad (2)$$

$$u = 0, \qquad r, \theta \in \partial M_{\text{top}} \qquad (3)$$

$$u(r, \theta, 0) = 0, \qquad r, \theta \in M \qquad (4)$$

$$u_t(r, \theta, 0) = g(r, \theta), \qquad r, \theta \in M\,. \qquad (5)$$

Problem 8

Consider a chain pinned at $x = L$ and such that its bottom end can move freely along the horizontal line $x = 0$. Thus, x is the vertical coordinate, and let the horizontal displacement of such a chain from the vertical line be denoted u. Separation of variables for the equation governing u leads to the following BVP:

$$xu'' + u' + \lambda^2 u = 0, \qquad x > 0 \qquad (1a)$$

$$u(L) = 0\,, \qquad (1b)$$

$$u(0) \text{ is bounded,} \qquad (1c)$$

and the prime denotes d/dx. In $(1a)$, λ^2 is the eigenvalue of the operator $-\left(x\dfrac{d^2}{dx^2} + \dfrac{d}{dx}\right)$ subject to the boundary conditions $(1b, c)$.

(a) Verify that $u(s)$, where $s = 2\sqrt{x}$, satisfies the equation

$$\frac{d^2u}{ds^2} + \frac{1}{s}\frac{du}{ds} + \lambda^2 u = 0 \qquad (2)$$

and relate the solution $u(s)$ of (2) to a Bessel function.

(b) Put Eq. $(1a)$ (not (2)!) in the standard Sturm-Liouville form. Then derive an orthogonality relation for two eigenfunctions $u_\lambda(x)$ and $u_\mu(x)$ corresponding to *different* eigenvalues λ^2 and μ^2. Make sure to correctly determine the weight in this orthogonality relation.