UVM Physics MS: Comprehensive Exam

Date: Saturday January 17, 2015
Time: 9:00 AM - 1:00 PM

INSTRUCTIONS

- **Provide careful and detailed solutions to 4 out of the 5 problems, starting each on a new page.**
- Only write your name on the first page of the exam booklet, but keep all your solutions together.
- The first three problems in Classical Mechanics, Electricity & Magnetism and Quantum Mechanics are **mandatory**. You must answer **one** problem in *either* Thermal/Statistical Physics or Mathematical Physics.
- You may attempt all 5 problems, but you must indicate which you would like to be graded by circling their numbers below.
- The exam is closed book; any formulas you may need will be provided.

	Question	Points	Score
	1	10	
	2	10	
40	3	10	
	4	10	
	5	10	
	Total:	50	

Name: ______________________________

1. [10 points] **Classical Mechanics**

Consider a particle with mass m moving in a plane, under the influence of the force:

$$\mathbf{F} = -k\mathbf{r}, \quad k > 0.$$

Here $\mathbf{r} = (x, y)$ is the two-dimensional position of the particle relative to the origin. This force corresponds to a two-dimensional isotropic harmonic oscillator.

(a) Is the total energy E conserved?

(b) Write down the Lagrangian in polar coordinates (r and ϕ). (You have to express both the kinetic and the potential energy in terms of r, ϕ.)

(c) Write down the two Lagrange equations, corresponding to the two polar variables (r and ϕ). Do not solve them.

(d) Write down the z component of the angular momentum. Is it conserved? (this should follow from one of the Lagrange equations).

2. [10 points] **Electricity & Magnetism**

A straight metal wire with circular cross section (area A) and conductivity σ carries a steady uniform current density J.

(a) Find the electric field $\mathbf{E}$ inside the wire.

(b) Determine the direction and magnitude of the Poynting vector $\mathbf{S}$ at the surface of the wire.

(c) Calculate the flux of $\mathbf{S}$ over a surface containing a segment of the wire length L. Compare this result to the Joule heat produced by this segment of wire.

3. [10 points] **Quantum Mechanics**

(a) A particle in a one dimensional box of side L is in its ground state for all time $t < 0$ when suddenly at $t = 0$ the size of the box increases to $2L$. How likely is it that the particle ends up in the first excited state of the new box for $t > 0$? Hint: drawing some pictures might be helpful.

(b) The Hamiltonian of the one-dimensional simple harmonic oscillator with mass m and spring constant $k = m\omega^2$ is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

Using the fact that $|n\rangle$ is an eigenstate of the Hamiltonian with energy $E_n = \hbar\omega(n + 1/2)$, derive the ladder operators $a^\dagger$ and a from the time independent Schrödinger equation where the number operator is defined to be $\hat{n} = a^\dagger a$ such that $\hat{n}|n\rangle = n|n\rangle$.

(c) Compute the commutator $[a, a^\dagger]$. What does this tell you about the statistics of the ladder operators?

4. [10 points] **Thermal/Statistical Physics**

Consider a 3-D oscillator; its energies are given as:

$$\varepsilon = n\hbar\omega - \varepsilon_0,$$

with $n^2 = n_x^2 + n_y^2 + n_z^2$, where n_x, n_y, n_z range from zero to infinity and ε_0 is a positive constant.

(a) Write down the general formula for the partition function in terms of the energy levels of the system.

(b) Calculate the partition function Z_s for this oscillator.

(c) Find the free energy F_s for the oscillator.

(d) Now consider ideal gas system of a cube of side length L and volume $V = L^3$. The system has N particles with each particle of mass M and energies as given below:

$$\varepsilon = \frac{\hbar^2}{2M}\left(\frac{\pi}{L}\right)^2 n^2 = \alpha n^2,$$

also with $n^2 = n_x^2 + n_y^2 + n_z^2$, where n_x, n_y, n_z range from zero to infinity. Find the partition function, the free energy and the chemical potential of the system. (Use the Sterling approximation for large N, $\ln N! = N \ln N - N$.)

5. [10 points] **Mathematical Physics**

Use the residue theorem for integration in the complex plane to evaluate the following integrals. Show clearly the contour of integration you are using.

(a) First, state the residue theorem, i.e. the formula that relates the integral of a function over closed contour in the complex plane and the residues at the poles.

(b)
$$\int_0^\infty \frac{dy}{1+y^2} = ?$$

(c)
$$\int_0^\infty \frac{\sin x}{x} dx = ?$$

You have to choose a contour to go around the pole (either above or below).

(d)
$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2} = ?$$

Notice that it is not a simple pole in this case.

UVM Physics MS: Comprehensive Exam

Date:	Friday January 22, 2016
Time:	8:00 AM - 12:00 PM

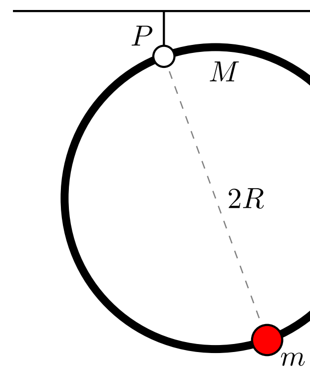
INSTRUCTIONS

- **Provide careful and detailed solutions to 4 out of the 5 problems, starting each on a new page.**
- Only write your name on the first page of the exam booklet, but keep all your solutions together.
- The first three problems in Classical Mechanics, Electricity & Magnetism and Quantum Mechanics are **mandatory**. You must answer **one** problem in *either* Thermal/Statistical Physics or Mathematical Physics.
- You may attempt all 5 problems, but you must indicate which you would like to be graded by circling their numbers below.
- The exam is closed book; any formulas you may need will be provided.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	40	

Name: ____

1. [10 points] **Classical Mechanics**



A thin hoop of radius R and mass M oscillates in its own plane with one point of the hoop fixed to a frictionless pivot at point P. The moment of inertia for such a hoop is $I = 2MR^2$. Attached to the hoop is a small bead of mass m. Consider only small oscillations of the bead-hoop system.

(a) At time $t = 0$ the fixed ring-bead system is displaced by a small angle $\phi_0 \ll 1$. Find the equation of motion $\phi(t)$ for the bead. What is the period τ_0 of the oscillatory motion? Describe why such motion is considered to be "harmonic." Why is this type of motion so ubiquitous in nature.

(b) Now suppose the small mass m is unstuck and allowed to slide freely along the hoop in a frictionless manner. Find the natural (eigen) frequencies of the bead's oscillation and draw diagrams that fully characterize their motion.

2. [10 points] **Electricity & Magnetism**

Consider a steady current I, uniformly distributed over the *surface* of an infinitely long cylinder with radius R, and flowing parallel to the symmetry axis of the cylinder (let us call it z direction). Let us use cylindrical coordinates: (ρ, φ, z), where ρ is the polar distance in the plane perpendicular to the cylinder axis, and φ is the polar angle in the plane. $(\hat{\rho}, \hat{\varphi}, \hat{z})$ are the corresponding unit vectors.

It is convenient to use Ampere's law in this case to find the magnetic field produced by the current.

(a) State Ampere's law which relates the circulation of the magnetic field $\mathbf{B}$ around a given closed curve, and the electric current.

(b) Find the magnetic field $\mathbf{B}(\rho)$, magnitude and direction, inside ($\rho < R$) and outside ($\rho > R$) the cylinder.

Make a plot (by hand) of the magnitude of the field $|\mathbf{B}(\rho)|$ as a function of the radial distance in the whole range $0 < \rho < \infty$.

(c) Find the vector potential $\mathbf{A}(\rho)$ that corresponds to the magnetic field from part (b), both inside and outside the cylinder.

Hint(s): You can use the fact that, by symmetry, the vector potential is along the current (z) direction, i.e. $\mathbf{A}(\rho) = A_z(\rho)\hat{z}$, and of course $\nabla \times \mathbf{A} = \mathbf{B}$. The curl operator has a very simple differential representation for a problem with our symmetry: $\nabla \times \mathbf{A} = -\frac{\partial A_z(\rho)}{\partial \rho}\hat{\varphi}$.

(d) Show that the divergence of the vector potential from (c) is zero, i.e. $\nabla \cdot \mathbf{A} = 0$. (This is a common condition imposed on $\mathbf{A}$ and should be automatically satisfied for a problem with cylindrical symmetry.)

3. [10 points] **Quantum Mechanics**

Consider a quantum particle with energy $E > 0$ and mass m moving (from left to right) in a one-dimensional potential $U(x)$ defined as: $U(x) = 0, x < 0; U(x) = U_0 > 0, x > 0$, which is simply a potential step. Consider (in parts (a,b,c)) the energy range $E \geq U_0$, i.e. the particle is at the top or above the potential step.

(a) Write down the Schrodinger equation and show that the wave-functions can be chosen to be: $\psi_<(x) = e^{ik_1x} + Be^{-ik_1x}, \quad x < 0$ (incoming and reflected wave), and $\psi_>(x) = Ae^{ik_2x}, \quad x > 0$ (transmitted wave). A, B are constants, yet to be determined. Write the momenta as a function of energy, $k_{1,2}(E)$.

(b) Find the coefficient B and from there the reflection coefficient (which is a function of energy), $R(E)$, naturally defined as $R = |B|^2$, in terms of $k_{1,2}(E)$.

(c) Consider the following limiting cases for the reflection $R(E)$: (1.) Let the energy be $E \to U_0$ (i.e. close to the edge of classical transmission), find $R \to$?, and (2.) Let the energy be very high, $E \gg U_0$, what is the asymptotic behavior, $R \to$?

Do the results make sense? (provide any discussion you feel is needed)

(d) Finally, let $E < U_0$. What is the value of the reflection, $R =$?, in this case (should follow immediately from the formulas and also make sense physically).

Taking everything learned so far into account, make a sketch (by hand) of the overall behavior of the function $R(E)$ in the whole energy range $0 < E < \infty$.

4. [10 points] **Thermal/Statistical Physics**

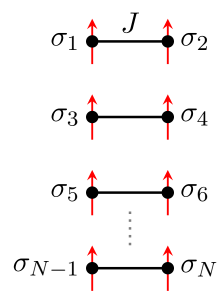
Paramagnetism describes the tendency of spins to line up in the direction of an applied magnetic field. In this problem you will investigate how interactions between spins can enhance or inhibit this behavior.

(a) Consider a single (Ising) spin described by a classical variable $\sigma = \pm 1$. It interacts with an external magnetic field of strength h with Hamiltonian:

$$H = -h\sigma.$$

Calculate the partition function, free energy and magnetization $m = \langle \sigma \rangle$ as a function of the magnetic field h and temperature T. This is the simplest example of paramagnetism.

(b) Now consider how interactions affect this picture by considering the "spin-ladder" shown in the diagram consisting of $N/2$ rungs for a total of N spins. Each spin interacts individually with the magnetic field h while pairs of spins interact with a strength J *only* across the rungs. Spins along the legs of the ladder remain independent. The Hamiltonian is modified to:



$$H = -J \sum_{i=1}^{N/2} \sigma_{2i-1}\sigma_{2i} - h \sum_{i=1}^{N} \sigma_i.$$

Discuss the behavior of the magnetization per spin:

$$m = \frac{1}{N} \left\langle \sum_{i=1}^{N} \sigma_i \right\rangle$$

for the ladder system as a function of βh when $J = 0$, and qualitatively when $\beta J \gg 1$ and $\beta J \ll -1$.

(c) Calculate the partition function, free energy and magnetization per spin *exactly* for the ladder system.

(d) Using your result, plot m vs. βJ for $\beta h \sim 1$. Include both signs of βJ. Show that your result is consistent with the discussion of part (b).

5. [10 points] **Mathematical Physics**

Use the residue theorem for integration in the complex plane to evaluate the following integrals. Show clearly the contour of integration you are using.

(a) First, state the residue theorem, i.e. the formula that relates the integral of a function over closed contour in the complex plane and the residues at the poles.

(b) Evaluate

$$I_1(a) = \int_{-\infty}^{\infty} dx \frac{e^{-iax}}{1+x^2} = ?, \quad a \text{ is real } \& \; a > 0.$$

Show clearly the contour of integration in the complex plane you are using. (You have to close the contour in the upper or lower half-plane; your choice should follow from the sign of a.)

(c) What is the limit of $I_1(a \to \infty) = ?$
Does your result make sense (or not)?

(d) Evaluate

$$I_2 = \int_0^{\infty} \frac{dx}{(1+x^4)} = ?$$

Please show clearly the contour of integration in the complex plane. (The result should be: $\pi/(2\sqrt{2})$, and we have also used $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$.)