UVM Physics MS: Comprehensive Exam

Date: Saturday January 11, 2013
Time: 8:00 AM - 12:00 PM

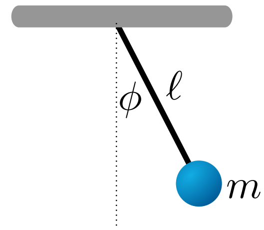
INSTRUCTIONS

- **Provide careful and detailed solutions to 4 out of the 5 problems, starting each on a new page.**
- Only write your name on the first page of the exam booklet, but keep all your solutions together.
- The first three problems in Classical Mechanics, Electricity & Magnetism and Quantum Mechanics are **mandatory**. You must answer **one** problem in *either* Thermal/Statistical Physics or Mathematical Physics.
- You may attempt all 5 problems, but you must indicate which you would like to be graded.
- The exam is closed book; any formulas you may need will be provided.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	40	

Name: ______________________________

1. [10 points] **Classical Mechanics**



Consider a simple pendulum as shown to the right, where a bob of mass m is hanging from a light string of length ℓ. At time $t = 0$ the mass is displaced by an angle ϕ_0.

(a) Find the equation of motion for $\phi(t)$ for the case where ϕ_0 is small. What is the period τ_0 of oscillatory motion? Describe why such motion is considered to be "harmonic." Why is this type of motion so ubiquitous in nature?

(b) Now consider the more interesting case where ϕ_0 is not small. Using conservation of energy, prove that the period of oscillations τ can be written in terms of the complete elliptic integral of the first kind K:

$$\frac{\tau}{\tau_0} = \frac{2}{\pi} K \left(\sin \frac{\phi_0}{2} \right)$$

where

$$K(k') = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2 t^2)}}$$

and τ_0 is the period for small angle oscillations in (a). Hint: you may find the trig identity $2\sin^2 A = 1 - \cos 2A$ useful when simplifying.

(c) Perform a Taylor expansion of the integrand to determine the correction to τ_0 (as a number) if $\phi_0 = \pi/4$.

2. [10 points] **Electricity & Magnetism**

A thin, very long cylindrical insulating shell of radius a carries a uniform surface charge density σ (charge per unit area).

(a) Find the electrostatic field everywhere in space.

(b) The shell is now rotating around its axis ($\hat{z}$-axis) with the frequency $\omega_0 = \text{const}$. The rotating insulator produces a surface current density. Find the magnetic field generated everywhere in space.

(c) After a while the cylinder starts to slow down at a constant rate *i.e.* $\omega(t) = \omega_0 - \alpha t$ where $\alpha \in \mathbb{R} > 0$. Find the electric field induced by the time dependent magnetic fields.

(d) Find the electromagnetic energy flow rate through the entire surface area of the cylinder.

3. [10 points] **Quantum Mechanics**

Consider a particle of mass m in one dimension (coordinate x), subjected to an attractive δ function potential $U(x) = -U_0\delta(x)$, $U_0 > 0$. It is known that there is one bound state solution in such a potential, with energy $E_0 < 0$.

(a) Write down the Schrödinger equation for the wave function in this potential. Choose the wave function in the regions $x > 0$ and $x < 0$, so that it is normalizable (exponentially decays at infinity.) Write down the normalization condition.

(b) Impose the correct boundary conditions at the origin. Show that in addition to the wave-function continuity equation $\psi(+0) = \psi(-0)$, there is also a condition on the wave-function derivative (which experiences a jump):

$$\psi'(+0) - \psi'(-0) = -\frac{2m}{\hbar^2}U_0\psi(0)$$

(c) From your previous results find E_0 in terms of U_0, m and $\hbar$.

4. [10 points] **Thermal/Statistical Physics**

The energy of free electrons can be written as:

$$\varepsilon = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$

where $n \in \mathbb{Z}^+$. Each electron has spin-1/2 and magnetic dipole moment μ. For a system consisting of N electrons:

(a) Find the density of states of electrons as a function of energy in one dimension where L is the length of the system.

(b) Find the density of states of electrons in two dimensions where the area of the system is $A = L^2$.

(c) Consider the two dimensional case at zero temperature and answer the following:

i. Find the Fermi energy ε_F. What does this number physically represent?

ii. In the absence of any external magnetic field, what is the number of spin up electrons?

iii. If there is an applied magnetic field B, the spin up direction is along the magnetic field. Under the condition $\mu B \ll \varepsilon_F$, what is the number of spin-up electrons? (Hint: the spin-up electron has lower magnetic potential energy, $u_{\text{up}} = -\mu B, u_{\text{down}} = \mu B$.)

5. [10 points] **Mathematical Physics**

(a) Evaluate the Fourier transform of a square pulse function

$$f(x) = \begin{cases} 1 & ; \quad |x| < a \\ 0 & ; \quad |x| > a \end{cases}.$$

(b) Parseval's theorem states

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk$$

where $F(k)$ is the Fourier transform of $f(x)$. Use this to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2} dy.$$

(c) Evaluate the integral in (b) using contour integration in the complex plane. (Be sure to sketch your contours clearly.)