MINIMAL HEILBRONN CHARACTERS OF FINITE GROUPS

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Abstract

Call a virtual character θ of a finite group G a *Heilbronn character* if its inner product with every monomial character of G is nonnegative. If moreover θ restricts to a character of every proper subgroup and quotient (where we define the restriction of θ to G/N as the sum of the constituents of θ whose kernels contain N), but is not a character of G itself, then θ is said to be *minimal*. Heilbronn characters arise naturally in the study of Artin's Conjecture on the holomorphy of L-series, where a hypothetical minimal counterexample engenders a corresponding minimal Heilbronn character of the associated Galois group. Although motivated originally by this number theoretic application, the study of Heilbronn characters is of independent interest to both group theory and representation theory.

A natural subclass of Heilbronn characters to classify are those that are both minimal and *unfaithful*, where θ is said to be unfaithful if the set $\{g \in G \mid \theta(g) = \theta(1)\}$ is nontrivial. Our main result establishes necessary and sufficient conditions for a finite group to possess an unfaithful minimal Heilbronn character. As an application of the main theorem we obtain a corollary bounding the p-rank of the Galois group of a minimal counterexample to Artin's Conjecture by the order of zero of a Dedekind zeta function.

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Chapter 1

Introduction

This dissertation describes the classification and construction of "unfaithful minimal Heilbronn characters" of finite groups, as well as their application to Artin's Conjecture on the holomorphy of L-series. In Sections 1.1 and 1.2 we present some basic background material and the definitions of the newer concepts considered herein. This allows us to state the main results in Section 1.3. Section 1.4 establishes the fundamental properties of Heilbronn characters. With these in hand, Section 1.5 completes the introduction by providing a number theoretic context from which these representation theoretic concepts evolved.

After collecting preliminary results in Chapter 2, the main theorems are proven in Chapters 3 and 4. In Chapter 5 we present a refinement of these results, as well as a general theorem pertaining to both faithful and unfaithful minimal Heilbronn characters. A complete characterization of the exceptional case of the main theorem is given in Chapter 6, and we conclude with a discussion of open questions and future research topics in Chapter 7.

We have attempted to confine the results from the literature to this introductory chapter and Section 2.1. Otherwise, except where explicitly stated, the results herein are more or less original. For example, in Section 2.2 we give our own proofs of well known results. Chapter 3 begins with a series of lemmas that are alluded to, if not explicitly stated, in [FF09], and then proceeds to diverge from that work. Similarly, the introductory lemmas for Chapter 4 include proofs of some known results (Lemma 4.2, though, is original). All of the remainder of the work is original, excepting Proposition 6.1 in Chapter 6 (which is again an original proof of a known result).

1.1 Background

We will assume here and throughout the body of this work that all groups mentioned are finite unless explicitly stated otherwise. Our notation is standard, as may be found in [DF04] or [CCN$^+$85].

1.1.1 Group Theory

Although we will assume familiarity with the basic results of elementary group theory as found in [DF04] or [Gor80], for the convenience of the reader we review here some of those concepts that are central to the statement of our theorems.

A group G is *perfect* if it is equal to its own commutator subgroup G', and a perfect group G is *quasisimple* if $G/Z(G)$ is a simple group. It is immediate that a quasisimple group is necessarily nonabelian.

If P is a p-group, $\Omega_1(P)$ is the subgroup generated by all of the elements of order precisely p. Of particular importance to the development of our results, if P is cyclic, then $\Omega_1(P)$ is the (unique) subgroup of order p.

The group $L_2(q)$, also written $PSL_2(q)$, is the *projective special linear group* of dimension 2 over the finite field of q elements. It is the quotient of the subgroup $SL_2(q)$ of determinant 1 matrices of the general linear group $GL_2(q)$ by its central subgroup of scalar matrices. When $q > 3$, $SL_2(q)$ is quasisimple and $L_2(q)$ is simple.

We will review other basic definitions throughout the text, as the need arises.

1.1.2 Representation Theory

We present here a brief overview of those aspects of representation and character theory that figure prominently in the body of this dissertation. Our aim is merely to introduce these topics and give the reader an informal sense of how they pertain, and our treatment will necessarily be both biased and terse. For a deeper and more balanced exposition the reader is encouraged to consult [DF04] or [Isa76].

A *representation* of a group G is a homomorphism from G into $GL(V)$, the group of nonsingular linear transformations of a vector space V. We will be concerned almost exclusively with *complex representations*, wherein V is a nonzero finite dimensional vector space over the complex numbers, and we will assume hereafter that this is the case unless otherwise specified. The *degree* of a representation is the dimension of the vector space V. A representation that fixes a nontrivial proper subspace of V is said to be *reducible*; one that does not is *irreducible*. By Maschke's Theorem a reducible (complex) representation decomposes the vector space V into a direct sum of invariant subspaces. These correspond to representations of smaller degree whose sum is the original representation, and in this way every representation can be decomposed into irreducible *constituents* in an essentially unique way.

Choosing a basis for V, we may assume that a given representation $\varphi : G \to GL(V)$ is a homomorphism from G into the matrix group $GL_n(\mathbb{C})$ (where n is the dimension of V over $\mathbb{C}$), since $GL(V) \cong GL_n(\mathbb{C})$. The function $\chi : G \to \mathbb{C}$ given by assigning to each element $g \in G$ the trace (sum of the diagonal entries) of the matrix $\varphi(g)$ is the *character* afforded by the representation. It is easily seen that similar matrices have the same trace, hence the character is independent of the choice of basis. For the same reason characters are *class functions* — they are constant on members of the same conjugacy class. We refer to characters as reducible or irreducible according to whether the representations affording them are reducible or irreducible. It can be shown that every finite group G has a well-defined set $\mathrm{Irr}(G)$ of irreducible characters, and that these form a basis for the space of all complex class functions on G. We define an inner product on this space by

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}. \tag{1.1}$$

The inner product has the property that if f is a class function on G with $f = a_1\chi_1 + \cdots + a_r\chi_r$ its decomposition into irreducible constituents, then $\langle f, \chi_i\rangle = a_i$.

Let H be a subgroup of G. For any character χ of G, we obtain a character of H by *restricting* χ to H: $\chi|_H(h) = \chi(h)$ for all $h \in H$. The "reverse" construction is more complicated, and in general a character ψ of H cannot be extended to a character of G. We can, however, obtain a character of G by *inducing* ψ from H to G:

$$\mathrm{Ind}_H^G(\psi)(g) = \frac{1}{|H|}\sum_{\substack{x\in G;\\ x^{-1}gx\in H}} \psi(x^{-1}gx). \tag{1.2}$$

We will often use the abbreviation ψ^* for the more cumbersome $\mathrm{Ind}_H^G(\psi)$. The operations of restriction and induction are related by *Frobenius Reciprocity*, which states that for any characters χ of G and ψ of H,

$$\langle \chi|_H, \psi\rangle_H = \langle \chi, \psi^*\rangle_G. \tag{1.3}$$

Here the subscripts H and G emphasize the group in which the inner product is calculated. Since this is readily deduced, we will usually omit the subscript.

If χ is a character of G, then $\chi(1)$ is the *degree* of χ, which is easily seen to equal the degree of the affording representation. We call a degree 1 character *linear*, and a character induced from a linear character of some subgroup *monomial*. A group whose irreducible characters are all monomial is called an *M-group*.

Although the sum and product of two characters is again a character, the difference of two characters, in general, is not. We may, however, form the ring of *generalized* or *virtual* characters by taking all $\mathbb{Z}$-linear combinations of characters of G. This is the setting of our dissertation — the "Heilbronn characters" of a group G are those virtual characters whose monomial constituents occur with nonnegative multiplicity. We elaborate in the next section.

1.2 Definitions

Definition 1. *A* ***Heilbronn character*** *of a finite group G is a virtual character θ of G whose inner product $\langle\theta, \mu\rangle$ is nonnegative for every monomial character μ of G.*

Equivalently, the inner product $\langle\theta|_H, \lambda\rangle$ is nonnegative for every linear character λ of every subgroup H of G (we will establish this equivalence formally in Lemma 1.1).

Observe that every character of a group G is as well a Heilbronn character of G. We will be interested in Heilbronn characters that are *not* characters. Before defining these "minimal" Heilbronn characters, we first formalize the notion of the restriction of an arbitrary virtual character to a quotient:

Definition 2. *If N is a normal subgroup and θ a virtual character of the finite group G, define $\theta|_{G/N}$, the "restriction" of θ to G/N, as:*

$$\theta|_{G/N}(gN) = \sum_{\substack{\chi \in Irr G; \\ N \leq \ker \chi}} \langle \theta, \chi \rangle \chi(g)$$

In other words, $\theta|_{G/N}$ is the sum of the irreducible constituents of θ whose kernels contain N, each taken according to its multiplicity in θ. It follows from the correspondence between characters of G/N and characters of G having N in their kernels (cf. Lemma 2.24) that $\theta|_{G/N}$ is a virtual character of G/N.

Definition 3. *A* ***minimal Heilbronn character*** *of a finite group G is a Heilbronn character that is not a character of G but whose restriction to every proper subgroup and every proper quotient group is a character of the subgroup or quotient.*

When θ is a minimal Heilbronn character of G and H is a proper subgroup of G, we may refer to the *kernel* of $\theta|_H$ since $\theta|_H$ is the character of some representation Ψ of H. It is easily seen that

$$\ker \Psi = \{g \in H \mid \theta|_H(g) = \theta|_H(1)\}. \tag{1.4}$$

Thus $\theta|_H$ is faithful precisely when the set in (1.4) is trivial.

It is convenient to use this observation to extend the usual definitions of "faithful" and "unfaithful" to virtual characters, even though the meaning in the sense of isomorphism to the image of a mapping is lost (since a virtual character does not in general have a corresponding representation).

Definition 4. *A virtual character θ of a group G is said to be* ***faithful*** *if and only if the set*

$$\{g \in G \mid \theta(g) = \theta(1)\} \tag{1.5}$$

consists solely of the identity. A virtual character that is not faithful is said to be ***unfaithful****.*

Note especially that for a virtual character of G that is not a character, the set in (1.5) need not be a subgroup of G (it is in general not closed under the group operation), although it is a union of G-conjugacy classes. If θ is a minimal Heilbronn character, however, the intersection of this set with every proper subgroup H of G is a normal subgroup of H. This observation is recorded formally in Lemma 1.5, and will be used extensively in the proof of our main theorem.

1.3 Statement of Results

Our main theorem characterizes the groups possessing unfaithful minimal Heilbronn characters:

Theorem 1. *Suppose G is a finite group with an unfaithful minimal Heilbronn character θ. Then θ restricts to an unfaithful character of some Sylow subgroup of G, and if P is a Sylow p-subgroup of G on which θ is unfaithful, then all of the following hold:*

(i) p is odd,

(ii) G is quasisimple with a cyclic center of order prime to p,

(iii) P is cyclic,

(iv) $N_G(P)$ is a maximal subgroup of G, and

(v) Either $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$, or $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$. (In the latter case $\Omega_1(P)$ is also contained in a Borel subgroup $N_G(Q)$ for some Sylow q-subgroup Q of G.)

Conversely, suppose G is a finite group and for some prime p and Sylow p-subgroup P of G conditions (i)–(v) above hold. Then G has an unfaithful minimal Heilbronn character that restricts to an unfaithful character of P.

Since an elementary abelian p-group of rank r does not have a faithful complex representation of degree less than r, if a minimal Heilbronn character θ is faithful on a Sylow p-subgroup P, then the p-rank of G is at most $\theta(1)$. If instead θ is unfaithful on P, then P is cyclic by Theorem 1, so the p-rank of G is 1. Hence in any case:

Corollary 1. *If θ is a minimal Heilbronn character of the finite group G, then the p-rank of G is at most $\theta(1)$ for any prime p dividing the order of G.*

We will see in Section 1.5 that minimal counterexamples to Artin's Conjecture give rise to minimal Heilbronn characters of the associated Galois groups. In this context the degree of the minimal Heilbronn character is the order of zero of a Dedekind zeta function at a pole of the L-series. Hence Corollary 1 implies:

Corollary 2. *Let E/F be a finite Galois extension of number fields with Galois group G. If the order of zero of the Dedekind zeta function $\zeta_E(s)$ at $s = s_0$ is strictly less than the p-rank of G for some prime p, then E/F is not a minimal counterexample to Artin's Conjecture at s_0.*

Moreover, a counterexample to Artin's Conjecture that is not minimal gives rise to a minimal Heilbronn character of some *section* of the associated Galois group, where H/K is a section of G for any $1 \leq K \trianglelefteq H \leq G$. Observing that if $E_1 \subseteq E$ then the order of zero of $\zeta_{E_1}(s)$ at $s = s_0$ is at most the order of zero of $\zeta_E(s)$ at $s = s_0$ (by the Aramata - Brauer Theorem that $\zeta_E(s)/\zeta_{E_1}(s)$ is holomorphic — cf. [Bra47]), Corollary 2 can be restated in terms of the sectional p-rank (the maximal p-rank of a section) and an arbitrary (not necessarily minimal) counterexample:

Corollary 2′. *Let E/F be a finite Galois extension of number fields with Galois group G. If the order of zero of the Dedekind zeta function $\zeta_E(s)$ at $s = s_0$ is strictly less than the sectional p-rank of G for some prime p, then E/F is not a counterexample to Artin's Conjecture at s_0.*

In Chapter 3 we prove that a group with an unfaithful minimal Heilbronn character has the properties stated in Theorem 1, and we complete the proof of Theorem 1 in Chapter 4 by constructing an unfaithful minimal Heilbronn character for an arbitrary group having these properties. In fact we will show that for such a group we have a great deal of control over the kernel of the Heilbronn character, strengthening Theorem 1 as follows:

Theorem 2. *Let P be a Sylow p-subgroup of the finite group G, and suppose G has a minimal Heilbronn character that is unfaithful on P. Then if P_1 is any nontrivial subgroup of P, G has a minimal Heilbronn character θ with $\ker \theta|_P = P_1$.*

The characterization in Theorem 1 of groups possessing unfaithful minimal Heilbronn characters reduces the explicit determination of such groups to questions about maximal subgroups of simple groups. Specifically, except when $G \cong L_2(q)$, $N_G(P)$ must be maximal and p cannot divide the order of any maximal subgroup not conjugate to $N_G(P)$. Work by many researchers on the maximal subgroups of simple groups, as discussed in Section 5.1, allows us to significantly restrict the possibilities for groups with unfaithful minimal Heilbronn characters. We state below a general theorem indicating the scope of these restrictions on the families of quasisimple groups. A more technical version of this result is given at the outset of Section 5.1.

Theorem 3. *If G is a finite group possessing an unfaithful minimal Heilbronn character, then G is quasisimple and $G/Z(G)$ is isomorphic to one of the following:*

(1) an alternating group of prime degree,

(2) one of the sporadic simple groups J_1, M_{23}, Ly, J_4, Fi'_{24}, or B,

(3) $L_n(q)$, $U_n(q)$, or $PSp_{2n}(q)$, with n prime in each case, or

(4) a member of one of the remaining families of groups of Lie type: an orthogonal group, a Suzuki group, a Ree group, ${}^3D_4(q)$, $G_2(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$.

Section 5.1 includes as well some discussion of when these necessary conditions are also sufficient.

Finally, in Section 5.2 we prove the following theorem characterizing those groups possessing a minimal Heilbronn character, without regard to whether or not the minimal Heilbronn character is faithful:

Theorem 4. *A finite group G possesses a minimal Heilbronn character if and only if there exists a faithful, irreducible, nonlinear, primitive character χ of G such that whenever χ is a constituent of ψ^* for some irreducible character ψ of a maximal subgroup of G, some irreducible character $\chi' \neq \chi$ of G is also a constituent of ψ^*.*

1.4 Basic Properties of Heilbronn Characters

We establish in this section some of the basic properties of Heilbronn characters. We begin with equivalent forms of Definition 1 and Definition 3:

Lemma 1.1. *A virtual character θ of G is a Heilbronn character if and only if the inner product $\langle \theta|_H, \lambda \rangle$ is nonnegative for every linear character λ of every subgroup H of G.*

Proof. Apply Frobenius Reciprocity to Definition 1. $\square$

Lemma 1.2. *A Heilbronn character θ of G is a minimal Heilbronn character of G if and only if θ is not a character of G, $\theta|_H$ is a character of H for every proper subgroup H of G, and $\langle \theta, \chi \rangle \geq 0$ for every unfaithful irreducible character χ of G.*

Proof. The first two conditions are unchanged from Definition 3; the third follows immediately from the definition of the restriction of θ to a quotient group (Definition 2): θ restricts to a character of every proper quotient of G if and only if $\langle \theta, \chi \rangle \geq 0$ for every unfaithful $\chi \in \mathrm{Irr}(G)$. $\square$

Since the zero function from G to $\mathbb{C}$ is a virtual character (consider, for example, $0 = \chi - \chi$), zero is a valid Heilbronn character by Definition 1 (observe, though, that if G is any finite group other than the trivial group, then zero is not a valid minimal Heilbronn character of G — its restriction to $1 < G$ is not a character). This explains the reference to a "nonzero" Heilbronn character in the following lemma.

Lemma 1.3. *Suppose θ is a Heilbronn character of G. Then:*

(i) If $H \leq G$, then $\theta|_H$ is a Heilbronn character of H.

(ii) If $N \trianglelefteq G$, then $\theta|_{G/N}$ is a Heilbronn character of G/N.

(iii) If a subgroup H of G is an M-group and θ is nonzero, then $\theta|_H$ is a character of H (this applies, in particular, when H is nilpotent).

Proof. In all three cases it is clear that θ restricts to a virtual character of the object in question. If ψ is a monomial character of $H \leq G$, then by the transitivity of induction $\mathrm{Ind}_H^G(\psi) = \psi^*$ is a monomial character of G. Hence by Frobenius Reciprocity $\langle \theta|_H, \psi \rangle = \langle \theta, \psi^* \rangle$, which is nonnegative, proving (i). It follows immediately that the restriction of θ to an M-group is a character, which is (iii).

Suppose now that $N \trianglelefteq G$, and let $\widehat{\mu}$ be a monomial character of G/N, induced from a linear character $\widehat{\lambda}$ of some subgroup H/N. By Frobenius Reciprocity $\langle \theta|_{G/N}, \widehat{\mu} \rangle = \langle \theta|_{H/N}, \widehat{\lambda} \rangle$, and by Definition 2 and Lemma 2.24,

$$\begin{aligned} \theta|_{H/N}(hN) &= \sum_{\substack{\psi \in \mathrm{Irr}(H); \\ N \leq \ker \psi}} \langle \theta|_H, \psi \rangle \psi(h) \\ &= \sum_{\widehat{\psi} \in \mathrm{Irr}(H/N)} \langle \theta|_H, \psi \rangle \widehat{\psi}(hN). \end{aligned}$$

In particular, since $\widehat{\lambda}$ is a linear character of H/N corresponding to the linear character λ of H whose kernel contains N, $\langle \theta|_{H/N}, \widehat{\lambda} \rangle = \langle \theta|_H, \lambda \rangle$ is nonnegative by part (i). This establishes (ii), completing the proof. □

It follows from Definition 3 and part (iii) of Lemma 1.3 that:

Lemma 1.4. *If G possesses a minimal Heilbronn character θ, then G is not an M-group. In particular, G is not nilpotent.*

Proof. Otherwise θ is a character of G, contrary to assumption. □

Lemma 1.5. *If θ is a minimal Heilbronn character of G and H is a proper subgroup of G, then $\ker \theta|_H \trianglelefteq H$.*

Proof. This follows from Definition 3 since $\theta|_H$ is a character of H. □

Minimal Heilbronn characters have the following properties:

Lemma 1.6. *Let θ be a minimal Heilbronn character of G. Then $\langle \theta, \chi \rangle < 0$ for some irreducible character χ of G, and every such irreducible character χ is faithful, nonlinear, and primitive.*

Proof. Since θ is a virtual character but not a character of G, $\langle \theta, \chi \rangle < 0$ for some irreducible character χ.

If χ is linear then in fact χ is monomial ("induced" from G to G), whence $\langle \theta, \chi \rangle < 0$ is a contradiction. Thus χ is nonlinear.

If χ is induced from a character ψ of a proper subgroup H of G, then by Frobenius Reciprocity $\langle \theta|_H, \psi \rangle < 0$. Thus $\theta|_H$ is not a character of H, a contradiction. Hence χ is primitive.

Suppose, finally, that χ is unfaithful, and let $K = \ker \chi$. Then $K \trianglelefteq G$ and χ corresponds to a character $\widehat{\chi}$ of G/K by Lemma 2.24. Moreover $\langle \theta|_{G/K}, \widehat{\chi} \rangle = \langle \theta, \chi \rangle < 0$ (by Definition 2), so $\theta|_{G/K}$ is not a character of G/K, a contradiction. Hence χ is faithful, completing the proof. $\square$

1.5 Number Theoretic Context and General Motivation

The original motivation for the study of Heilbronn characters comes from the investigation of the holomorphy of Artin L-series. In Section 1.5.1 we review the definition of these L-series and state Artin's Conjecture, and then proceed to describe the importance and current status of this conjecture. We establish the relationship between Artin's Conjecture and Heilbronn characters in Section 1.5.2. Finally, we survey the results leading up to the present paper in Section 1.5.3. Our general references for this section are [Hei67], [MM97], [Sny02], and [Foo97].

1.5.1 Artin's Conjecture

Let E/F be a finite Galois extension of number fields with Galois group G. Let $\mathcal{O}_E$ and $\mathcal{O}_F$ denote the rings of integers in E and F, respectively, and let $\mathfrak{p}$ be any prime in $\mathcal{O}_F$ with $\mathfrak{P}$ a prime in $\mathcal{O}_E$ lying over $\mathfrak{p}$ and $\mathfrak{p} \cap \mathbb{Z} = (p)$.

Recall the definitions of the *decomposition* and *inertia* groups of $\mathfrak{P}$, respectively:

$$G_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}, \text{ and}$$
$$I_{\mathfrak{P}} = \{\sigma \in G_{\mathfrak{P}} \mid \sigma(x) \equiv x \bmod \mathfrak{P} \quad \text{for all } x \in \mathcal{O}_E\}.$$

Thus $I_{\mathfrak{P}}$ is the kernel of the action of $G_{\mathfrak{P}}$ on $\mathcal{O}_E/\mathfrak{P}$, hence $I_{\mathfrak{P}} \trianglelefteq G_{\mathfrak{P}}$. Moreover since $\mathfrak{P}$ is a maximal ideal, $\mathcal{O}_E/\mathfrak{P}$ is a finite field and the quotient $G_{\mathfrak{P}}/I_{\mathfrak{P}}$ is isomorphic to a subgroup of its Galois group. In particular, $G_{\mathfrak{P}}/I_{\mathfrak{P}}$ is cyclic. There is an automorphism $Frob_{\mathfrak{P}} \in G_{\mathfrak{P}}$ that acts as the Frobenius automorphism $x \mapsto x^p$ for $x \in \mathcal{O}_E/\mathfrak{P}$; the coset of $I_{\mathfrak{P}}$ containing $Frob_{\mathfrak{P}}$ generates $G_{\mathfrak{P}}/I_{\mathfrak{P}}$.

Let V be a finite dimensional vector space over $\mathbb{C}$ and $T : G \to GL(V)$ a representation of G with corresponding character χ. Denote by 1_V the identity transformation in $GL(V)$. Define

$$V^{I_{\mathfrak{P}}} = \{v \in V \mid T(\sigma)(v) = v \quad \text{for all} \quad \sigma \in I_{\mathfrak{P}}\},$$

the set of vectors fixed by the (representation of the) inertia group of $\mathfrak{P}$. Recall that the *norm* of $\mathfrak{p}$ is

$$N_{F/\mathbb{Q}}(\mathfrak{p}) = |\mathcal{O}_F/\mathfrak{p}|\,,$$

i.e. the order of the finite field.

For any $s \in \mathbb{C}$, the *Artin L-series*, $L(s, \chi, E/F)$, is defined as:

$$L(s, \chi, E/F) = \prod_{\mathfrak{p}} \left[\det\left(1_V - N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\, T|_{V^{I_{\mathfrak{P}}}}(Frob_{\mathfrak{P}})\right)\right]^{-1} \tag{1.6}$$

where the product is taken over all primes $\mathfrak{p}$ in $\mathcal{O}_F$. It can be shown that this product is independent of the choice of $\mathfrak{P}$ lying over $\mathfrak{p}$ and the choice of coset representative for $Frob_{\mathfrak{P}}$, hence is well defined.

The Euler product defined in (1.6) converges for $\operatorname{Re}(s) > 1$, hence defines an analytic function in that half plane. By results of Artin, Hecke, and Brauer, this function can be continued to a meromorphic function of the entire complex plane. Artin's Conjecture is:

Conjecture 1.7. (E. Artin) *For any character χ of $Gal(E/F)$, the continuation of the L-series $L(s, \chi, E/F)$ defined in* (1.6) *is holomorphic on the entire complex plane except for a pole at $s = 1$ of order equal to the multiplicity of the principal character as a constituent of χ. In particular, if the principal character is not a constituent of χ, then the continuation is holomorphic on all of $\mathbb{C}$.*

When we wish to explicitly refer to the meromorphic continuation of the L-series, we will use the term *L-function*. Otherwise the term L-series will be used interchangeably for both the Euler product in (1.6) and its continuation.

The basic properties of Artin L-series are neatly presented in [Foo97] and reprised here. These form the foundation for our character theoretic treatment of Artin's Conjecture.

Proposition 1.8. *Let E/F be a finite Galois extension of number fields with Galois group G, let E^H denote the fixed field of H for any subgroup H of G, and for any character ψ of G let $L(s, \psi, E/F)$ denote the Artin L-series for ψ. Then:*

(1) $L(s, \psi_1 + \psi_2, E/F) = L(s, \psi_1, E/F)L(s, \psi_2, E/F)$*, where* ψ_1 *and* ψ_2 *are characters of* G*,*

(2) $L(s, \psi, E/F) = L(s, \widehat{\psi}, E^{\ker\psi}/F)$*, where* $\widehat{\psi}$ *is the character of* $G/\ker\psi$ *corresponding to* ψ *in the manner of Lemma* 2.24*,*

(3) $L(s, \xi, E/E^H) = L(s, Ind_H^G(\xi), E/F)$*, where* H *is a subgroup of* G *and* ξ *is a character of* H*, and*

(4) if λ *is a non-principal linear character of* G*,* $L(s, \lambda, E/F)$ *is entire; if* χ_0 *is the principal character of* G*,* $L(s, \chi_0, E/F) = \zeta_F(s)$ *is analytic everywhere except for a simple pole at* $s = 1$ *(here* $\zeta_F(s)$ *is the Dedekind zeta-function of* F*).*

By property (*4*), Artin's Conjecture is true for linear characters, and property (*3*) then implies that Artin's Conjecture holds as well for monomial characters. Thus Artin's Conjecture holds for all M-groups. Other than these, Artin's Conjecture is only known to hold for some specific characters of specific groups. Most notable among these are the degree two characters of S_4, $SL_2(3)$, and $GL_2(3)$ (these last two are two-fold covers of A_4 and S_4, respectively), which are known by work of Langlands and Tunnell [PY99, Section 4.1]. A case of particular interest that remains open is the degree two character of $SL_2(5)$, the two-fold cover of A_5. In Section 1.5.3 we discuss results relating Artin's Conjecture at a specific point $s_0 \in \mathbb{C}$ to the order of zero of a Dedekind zeta-function at s_0.

The importance of Artin's Conjecture is based on the relation of L-series to the distribution of primes in number fields. Certain results in this area would be significantly strengthened if Artin's Conjecture was known to hold — see for example the improved bounds on the effectively computable version of the Chebotarev Density Theorem in [MM97, pp. 46-52].

1.5.2 Heilbronn Characters Associated to Galois Extensions

The following virtual character, which provides the key to relating the holomorphy of Artin L-series to the study of Heilbronn characters, was first introduced by H. Heilbronn in [Hei73] (explaining the origin of the term *Heilbronn* character), and further developed by H. Stark [Sta74], R. Foote

[Foo97, Foo90], Foote and V. K. Murty [FM89], Foote and D. Wales [FW90], and S. L. Rhoades [Rho93] (also related is work by G. O. Michler [Mic94] and J. König [Kön09]).

Definition 5. *Let E/F be a finite Galois extension of number fields with Galois group G, and fix a point $s_0 \in \mathbb{C} - \{1\}$. Then*

$$\theta = \sum_{\chi \in Irr(G)} ord_{s=s_0} L(s, \chi, E/F) \cdot \chi$$

is the ***Heilbronn character associated to*** $\mathbf{E/F}$*, or just the* ***associated Heilbronn character*** *if the extension is understood.*

Here $\mathrm{ord}_{s=s_0} L(s, \chi, E/F)$ denotes the order of zero or pole of the meromorphic L-series at $s = s_0$ (and is a negative integer precisely when the L-series has a pole). We restrict our consideration to $\mathbb{C} - \{1\}$ since Artin's Conjecture is known to hold at $s = 1$.

We proceed to justify the terminology of Definition 5 and establish some of the basic properties of these objects.

Lemma 1.9. *Let E/F be a finite Galois extension of number fields with Galois group G, fix a point $s_0 \in \mathbb{C} - \{1\}$, and let θ be the associated Heilbronn character. For any subgroup H of G let E^H denote the fixed field of H. Then all of the following hold:*

(i) *θ is a Heilbronn character of G,*

(ii) *$\langle \theta, \psi \rangle = ord_{s=s_0} L(s, \psi, E/F)$ for any character ψ of G,*

(iii) *If $H \leq G$, then $\theta|_H$ is the Heilbronn character associated to the Galois extension E/E^H,*

(iv) *If $N \trianglelefteq G$, then $\theta|_{G/N}$ is the Heilbronn character associated to the Galois extension E^N/F,*

(v) *θ is an actual character of G or zero if and only if Artin's Conjecture is true at s_0,*

(vi) *$0 \leq \theta(1) = ord_{s=s_0} \zeta_E(s)$ where $\zeta_E(s)$ is the Dedekind zeta-function of E.*

Proof. We shall refer by number to the properties of Artin L-series in Proposition 1.8 (without explicitly citing the proposition).

We begin by proving (*ii*). Let $\psi = \sum_i \chi_i$ be the expression, unique up to order, of ψ as a sum of irreducibles. By property (*1*),

$$L(s, \psi, E/F) = \prod_i L(s, \chi_i, E/F),$$

so

$$\mathrm{ord}_{s=s_0} L(s, \psi, E/F) = \sum_i \mathrm{ord}_{s=s_0} L(s, \chi_i, E/F).$$

By Definition 5 this is $\sum_i \langle \theta, \chi_i \rangle$, which is $\langle \theta, \psi \rangle$. This proves (*ii*). Since Artin's Conjecture is true at s_0 if and only if $\mathrm{ord}_{s=s_0} L(s, \chi, E/F)$ is nonnegative for every irreducible character χ of G, (*v*) follows immediately.

We proceed to consider (*i*). Let μ be a monomial character of G induced from a linear character λ of a subgroup H. It follows from property (*3*) that

$$L(s, \mu, E/F) = L(s, \lambda, E/E^H),$$

which is analytic at s_0 by property (*4*). Hence the order of zero or pole of the L-series at $s = s_0$ is nonnegative. By (*ii*) this is precisely $\langle \theta, \mu \rangle$, which establishes (*i*).

For (*iii*), observe first that $H = Gal(E/E^H)$. Let ϕ be the Heilbronn character associated to E/E^H; we must show that $\theta|_H = \phi$. Let $\xi \in \mathrm{Irr}(H)$. We argue that

$$\begin{aligned} \langle \phi, \xi \rangle &= \mathrm{ord}_{s=s_0} L(s, \xi, E/E^H) && (1.7)\\ &= \mathrm{ord}_{s=s_0} L(s, \mathrm{Ind}_H^G(\xi), E/F) && (1.8)\\ &= \langle \theta, \mathrm{Ind}_H^G(\xi) \rangle && (1.9)\\ &= \langle \theta|_H, \xi \rangle. && (1.10) \end{aligned}$$

Specifically, (1.7) follows from (*ii*), (1.8) follows from property (*3*), (1.9) follows from (*ii*), and (1.10) follows from Frobenius Reciprocity. Hence $\theta|_H = \phi$ as claimed.

We prove (*iv*) similarly. Observe that $G/N = Gal(E^N/F)$, and let φ be the Heilbronn character associated to E^N/F. For $\widehat{\psi} \in \mathrm{Irr}(G/N)$, let ψ

denote the corresponding irreducible character of G having N in its kernel (cf. Lemma 2.24). We show that $\theta|_{G/N} = \varphi$.

From *(ii)*, $\langle \varphi, \widehat{\psi} \rangle = \text{ord}_{s=s_0} L(s, \widehat{\psi}, E^N/F)$. Our next step is to show that

$$L(s, \widehat{\psi}, E^N/F) = L(s, \psi, E/F). \tag{1.11}$$

Applying property *(2)* to $L(s, \widehat{\psi}, E^N/F)$ yields

$$L(s, \widehat{\psi}, E^N/F) = L(s, \widehat{\widehat{\psi}}, (E^N)^{\ker \psi}/F),$$

where $\widehat{\widehat{\psi}}$ is $\widehat{\psi}$ considered as a character of $G/\ker \psi$. Since $(E^N)^{\ker \psi} = E^{\ker \psi}$, property *(2)* implies (1.11), as required. Hence

$$\begin{aligned} \langle \varphi, \widehat{\psi} \rangle &= \text{ord}_{s=s_0} L(s, \widehat{\psi}, E^N/F) && (1.12) \\ &= \text{ord}_{s=s_0} L(s, \psi, E/F) && (1.13) \\ &= \langle \theta, \psi \rangle, && (1.14) \end{aligned}$$

where (1.12) follows from *(ii)*, (1.13) follows from (1.11), and (1.14) follows from *(ii)*. Since $N \leq \ker \psi$ by assumption, $\langle \theta, \psi \rangle = \langle \theta|_{G/N}, \psi \rangle$ by Definition 2. This establishes *(iv)*.

It remains to prove *(vi)*. Observe that $\theta(1) = \theta|_{\langle 1 \rangle}(1)$ and $\theta|_{\langle 1 \rangle}$ is the Heilbronn character associated to E/E by *(iii)*. Since the only irreducible character of $\langle 1 \rangle$ is the principal character χ_0, $\theta|_{\langle 1 \rangle} = \text{ord}_{s=s_0} L(s, \chi_0, E/E)$ by Definition 5, which is $\text{ord}_{s=s_0} \zeta_E(s)$ by property *(4)*. Since the Dedekind zeta-function is holomorphic at every $s_0 \neq 1$, $\theta(1) \geq 0$, completing the proof. □

Suppose E/F is a counterexample to Artin's Conjecture at s_0. Then there is some character χ of $Gal(E/F)$ such that $L(s, \chi, E/F)$ has a pole at s_0. In fact we may take χ to be an irreducible character, since otherwise the L-series of some irreducible constituent of χ is also a counterexample at s_0 (by property *(1)* of Proposition 1.8). We may further reduce to a *minimal* counterexample by applying properties *(2)* and *(3)* of Proposition 1.8. This motivates the following definition:

Definition 6. *Let E/F be a finite extension of number fields with Galois group G, and fix a point $s_0 \in \mathbb{C} - \{1\}$. Then E/F is a* ***minimal counterexample to Artin's Conjecture at*** $\mathbf{s_0}$ *if $L(s, \chi, E/F)$ has a pole at s_0 for some irreducible character χ of G, but $L(s, \psi, E_1/F_1)$ is holomorphic at s_0 for every irreducible character ψ of $Gal(E_1/F_1)$ for every proper Galois subextension E_1/F_1 of E/F (i.e. $F \subseteq F_1 \subseteq E_1 \subseteq E$ and $E_1/F_1 \neq E/F$).*

The key observation is:

Lemma 1.10. *If E/F is a minimal counterexample to Artin's Conjecture at s_0, then the Heilbronn character associated to E/F is a minimal Heilbronn character.*

Proof. Let θ be the associated Heilbronn character. By Lemma 1.9 parts (i) and (v), θ is a Heilbronn character but not an actual character of G. Let H be a proper subgroup and G/N a proper quotient of G (i.e. $N \neq 1$), and consider $\theta|_H$ and $\theta|_{G/N}$. By Lemma 1.9 parts (iii) and (iv), these are the Heilbronn characters associated to E/E^H and E^N/F, respectively. By assumption Artin's Conjecture holds at s_0 for these extensions since they are properly contained in E/F, hence the associated Heilbronn characters are actual characters by Lemma 1.9 part (v). It follows that θ is a minimal Heilbronn character by Definition 3. □

As a corollary, we obtain an equivalent form of Definition 6:

Corollary 1.11. *E/F is a minimal counterexample to Artin's Conjecture at s_0 if and only if all of the following hold:*

(i) $L(s, \chi, E/F)$ *has a pole at* s_0 *for some irreducible character* χ *of* G,

(ii) $L(s, \widehat{\psi}, E^N/F)$ *is holomorphic at* s_0 *for every character* $\widehat{\psi}$ *of every proper quotient* G/N *of* G, *and*

(iii) $L(s, \xi, E/E^H)$ *is holomorphic at* s_0 *for every character* ξ *of every proper subgroup* H *of* G.

Proof. If E/F is a minimal counterexample to Artin's Conjecture at s_0, then (i) is by assumption, and (ii) and (iii) follow from Lemma 1.10, Definition 3, and Lemma 1.9 parts (iii) and (iv).

Now suppose (i), (ii), and (iii) hold, and let E_1/F_1 be a proper Galois subextension of E/F. We must show that $L(s, \psi, E_1/F_1)$ is holomorphic at s_0 for every character ψ of $Gal(E_1/F_1)$, or equivalently that the Heilbronn character associated to E_1/F_1 is a character. Let H and N be subgroups of G such that $F_1 = E^H$ and $E_1 = E^N$. Then $\theta|_H$ is the Heilbronn character associated to E/E^H, and if H is proper in G then $\theta|_H$ is a character by condition (iii) and Lemma 1.9. Otherwise $H = G$ (so $F_1 = F$) and $1 \neq N \trianglelefteq G$ by the assumption that E_1/F_1 is Galois and not equal to E/F. Then $\theta|_{G/N}$

is the Heilbronn character associated to E^N/F, and $\theta|_{G/N}$ is a character by condition (*ii*) and Lemma 1.9. □

We conclude this section by emphasizing the results of Lemma 1.10: *minimal counterexamples to Artin's Conjecture give rise to minimal Heilbronn characters*. (The reverse is not true — we will see that many finite groups do indeed possess minimal Heilbronn characters; this does not imply the failure of Artin's Conjecture for extensions corresponding to these groups.)

1.5.3 Survey of Preceding Research

One of the primary motivations for the character theoretic investigation of Artin's Conjecture is a result of Stark [Sta74, Theorem 3]:

Theorem 1.12. (Stark) *Suppose $\theta(1) \leq 1$ where θ is the Heilbronn character associated to the Galois extension E/F at $s_0 \in \mathbb{C} - \{1\}$. Then Artin's Conjecture is true at s_0: $L(s, \chi, E/F)$ is holomorphic at s_0 for every character χ of Gal(E/F).*

Recall that $\theta(1)$ is the order of zero of the Dedekind zeta-function $\zeta_E(s)$ at $s = s_0$. Stark's Theorem tells us that "small order" zeros of $\zeta_E(s)$ force the holomorphy of *all* Artin L-series at those points. Foote and Murty were successful in further exploiting this approach by proving the following result [FM89]:

Theorem 1.13. (Foote – Murty) *Let E/F be a Galois extension of number fields with solvable Galois group G and let $p_1, p_2, \ldots, p_n$ be the distinct prime divisors of $|G|$ with $p_1 < p_2 < \cdots < p_n$. If the order of zero of $\zeta_E(s)$ at s_0 is at most $p_2 - 2$, then $L(s, \chi, E/F)$ is holomorphic at s_0 for every character χ of G.*

These successes inspired attempts to obtain more general results, in particular to prove theorems similar to Theorem 1.13 without assuming solvability. Since there are groups that are not solvable that do possess Heilbronn characters, the best one can hope for without additional number-theoretic information is a group theoretic characterization of the possible minimal counterexamples to Artin's Conjecture, parameterized in terms of $\theta(1)$. This was achieved by Foote and D. Wales for $\theta(1) = 2$ in [FW90]:

Theorem 1.14. (Foote – Wales) *Let E/F be a minimal counterexample to Artin's Conjecture at s_0 with associated minimal Heilbronn character θ, and assume that $\theta(1) \leq 2$. Then there is a Galois subextension E_1/F_1 of E/F (i.e. $F \subseteq F_1 \subseteq E_1 \subseteq E$) with Galois group isomorphic either to $SL_2(p)$ for some odd prime p, or to the extension of the quaternion group of order 8 by a cyclic 3-group. In particular if E/F has no such subextension, then $L(s, \chi, E/F)$ is holomorphic at s_0 for every character χ of $Gal(E/F)$.*

A critical case of Theorem 1.14 is when θ is an unfaithful minimal Heilbronn character. The situation when such a Heilbronn character is restricted to a Sylow 2-subgroup was treated by Foote in [Foo97] (and will reappear as Theorem 2.1 in Chapter 2):

Theorem 1.15. (Foote) *If θ is a minimal Heilbronn character of the finite group G, then θ is faithful on every Sylow 2-subgroup of G.*

This dissertation completes the investigation of unfaithful minimal Heilbronn characters by precisely characterizing the conditions under which a minimal Heilbronn character is unfaithful on a Sylow p-subgroup for odd primes p. The contrast between the odd prime case, in which unfaithful minimal Heilbronn characters do indeed occur, and the case for $p = 2$, in which they do not, is due in part to the greater complexity of the families of groups possessing strongly closed p-subgroups. On the other hand, we gain considerable advantage in the odd prime case from the established results for $p = 2$ by "switching primes" — involving elements of order 2 in proper subgroups generated by p-elements.

Ultimately, we cannot hope to prove Artin's Conjecture by these means alone — even for A_5, the smallest non-abelian simple group, Artin's Conjecture is not known to hold (and minimal Heilbronn characters of A_5 are easily constructed). Further constraints on the relation of the associated Heilbronn character to the irreducible characters of the Galois group are needed if we hope to verify Artin's Conjecture group theoretically, and such information must come from new number theoretic developments. It is hoped, however, that the results herein may be useful in identifying minimal obstructions to Artin's Conjecture, and may thereby provide some new insight or leverage, or even some weaker versions of the conjecture (such as bounding the size of potential poles).

Finally, it is important to observe that even if Artin's Conjecture was settled (by other means), the results on minimal Heilbronn characters in this paper and those preceding it remain important theorems of independent interest in representation theory.

Chapter 2

Preliminary Results

2.1 Assumed Results

We collect here those results that we assume without proof, excepting the basic results of elementary group and character theory, which we use freely without explicit reference. General references for such information include [DF04], [Gor80], and [Isa76]. We will occasionally refer directly to the *Atlas of Finite Groups* ([CCN$^+$85]) for details pertaining to specific groups.

The main theorem in [Foo97] is both the inspiration for our Theorem 1 and an indispensable tool in its proof. The theorem is:

Theorem 2.1. (Foote) *If θ is a minimal Heilbronn character of the finite group G, then θ is faithful on every Sylow 2-subgroup of G.*

The remainder of this section is organized into subsections of related results.

2.1.1 Finite Simple Groups

Many of our assumed results pertain to the finite simple groups in general; these we consider here.

We begin with the monumental Classification of the Finite Simple Groups:

Theorem 2.2. *Every finite simple group is cyclic of prime order, an alternating group, a finite simple group of Lie type, or one of the twenty-six sporadic finite simple groups.*

A revised proof of this result, which originally spanned literally hundreds of journal articles, is currently in progress; the first volume ([GLS94]) includes the above statement of the theorem. In addition to the proof of our main theorem, the assumed results in Theorem 2.1, Theorem 2.9, Proposition 2.10, and Table 2.6 all rely on the Classification of the Finite Simple Groups.

Although the statement in Theorem 2.2 is appealingly elegant, for practical purposes we require more explicit detail. This we address in Theorem 2.3 after first digressing to survey the groups of Lie type. Our reference for the following material is [GLS94].

The *classical* groups of Lie type, which were the first known families, consist of *linear*, *symplectic*, *unitary*, and *orthogonal* groups. These are matrix groups, each appearing in some form in a general linear group of invertible square matrices over a finite field. As an example, starting with the general linear group $GL_n(\mathbb{F}_q)$, the quotient of the subgroup of determinant 1 matrices by the central subgroup of scalar matrices is the *projective special linear* group $PSL_n(q)$, or just $L_n(q)$, which is generally simple. The symplectic, unitary, and orthogonal groups arise similarly, after first taking the subgroup of $GL_n(\mathbb{F}_q)$ preserving some binary form (in the unitary case the general linear group is taken over $\mathbb{F}_{q^2}$ instead of $\mathbb{F}_q$, and in the orthogonal cases the simple group is actually a subgroup of the projective quotient).

In 1955 Claude Chevalley described a procedure for obtaining finite groups as automorphism groups of simple Lie algebras. These *Chevalley groups* account for the linear, symplectic, and some orthogonal groups (with $L_{n+1}(q) \cong A_n(q)$, $PSp_{2n}(q) \cong C_n(q)$, $P\Omega_{2n+1}(q) \cong B_n(q)$, and $P\Omega^+_{2n}(q) \cong D_n(q)$), as well as others corresponding to the exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8 (the associated finite groups over fields of order q are designated $G_2(q)$, $F_4(q)$, etc.).

Chevalley's construction notably failed to produce the unitary groups and the full family of orthogonal groups. This deficiency was remedied by Robert Steinberg in 1959. To each simple Lie algebra, hence to each Chevalley group, there corresponds a *Dynkin diagram* — a graph that models the structure of a fundamental root system. Steinberg showed that automorphisms of the Dynkin diagram give rise to new families of finite groups. These *twisted* (or *Steinberg twisted*) Chevalley groups include the unitary groups as twisted versions of the linear groups ($U_{n+1}(q) \cong {}^2A_n(q)$, where the superscript denotes the order of the graph automorphism), the remaining orthogonal groups ($P\Omega^-_{2n}(q) \cong {}^2D_n(q)$), as well as two families with no classical analogues: ${}^3D_4(q)$ and ${}^2E_6(q)$. In light of Steinberg's results, the original Chevalley groups are often referred to as *untwisted*.

Finally, in 1961, Rimhak Ree showed how to exploit certain "exceptional" graph automorphisms (occurring over finite fields of specific characteristic) to produce new families of groups. This brought the *Suzuki groups*, discovered group-theoretically by Michio Suzuki, into the context of the groups of Lie type (with $Sz(2^{2n+1}) \cong {}^2B_2(2^{2n+1})$), and yielded as well the *Ree groups* ${}^2G_2(3^{2n+1})$ and ${}^2F_4(2^{2n+1})$.

We are now ready to elaborate on Theorem 2.2. We will take Theorem 2.3 to be our statement of the Classification of the Finite Simple Groups. The content is from [GLS94] and [CCN+85].

Theorem 2.3. *The finite simple groups are precisely:*

(1) the cyclic groups Z_p of prime order p;

(2) the alternating groups A_n of degree $n \geq 5$, of order $n!/2$;

(3) the finite simple groups of Lie type, whose orders are given in Table 2.2 (here q is a power of a prime):

- *(i) the linear groups $A_n(q) = PSL_{n+1}(q) = L_{n+1}(q)$, $n \geq 1$, excluding $A_1(2) \cong S_3$ and $A_1(3) \cong A_4$;*
- *(ii) the unitary groups ${}^2A_n(q) = PSU_{n+1}(q) = U_{n+1}(q)$, $n \geq 2$, excluding ${}^2A_2(2) \cong 3^2.Q_8$;*
- *(iii) the orthogonal groups of odd dimension $B_n(q) = P\Omega_{2n+1}(q) = \Omega_{2n+1}(q)$, $n \geq 2$, excluding $B_2(2) \cong S_6$;*
- *(iv) the Suzuki groups ${}^2B_2(q) = Sz(q)$, $q = 2^{2n+1}$, $n \geq 1$;*
- *(v) the symplectic groups $C_n(q) = PSp_{2n}(q)$, $n \geq 2$, excluding $C_2(2) \cong S_6$;*
- *(vi) the orthogonal groups of "plus" type in even dimension $D_n(q) = P\Omega^+_{2n}(q)$, $n \geq 3$;*
- *(vii) the orthogonal groups of "minus" type in even dimension ${}^2D_n(q) = P\Omega^-_{2n}(q)$, $n \geq 2$;*
- *(viii) ${}^3D_4(q)$;*
- *(ix) $G_2(q)$, excluding $G_2(2) \cong {}^2A_2(3).2$;*

(x) *the Ree groups* ${}^2G_2(q) = Re(q)$, $q = 3^{2n+1}$, $n \geq 1$;

(xi) $F_4(q)$;

(xii) *the Ree groups* ${}^2F_4(q) = Re(q)$, $q = 2^{2n+1}$, $n \geq 1$, *as well as the Tits group* ${}^2F_4(2)'$;

(xiii) $E_6(q)$;

(xiv) ${}^2E_6(q)$;

(xv) $E_7(q)$;

(xvi) $E_8(q)$;

(4) *the sporadic finite simple groups, whose orders are given in Table 2.3:*

(i) *the Mathieu groups* M_{11}, M_{12}, M_{22}, M_{23}, *and* M_{24};

(ii) *the Janko groups* J_1, J_2, J_3, *and* J_4;

(iii) *the Higman-Sims group* HS;

(iv) *the Held group* He;

(v) *the McLaughlin group* Mc;

(vi) *the Suzuki group Suz;*

(vii) *the Lyons group Ly;*

(viii) *the Rudvalis group Ru;*

(ix) *the O'Nan group* $O'N$;

(x) *the Conway groups* Co_1, Co_2, *and* Co_3;

(xi) *the Fischer groups* Fi_{22}, Fi_{23}, *and* Fi'_{24};

(xii) *the Harada-Norton group* HN;

(xiii) *the Thompson group* Th;

(xiv) *the Baby Monster* B;

(xv) *the Monster* M.

For each finite simple group there is a uniquely determined *universal cover* — a largest possible perfect central extension (for groups of Lie type, the simple group and its universal cover are referred to respectively as the *adjoint* and *universal* groups). The center of the universal cover of a simple group is called the *Schur multiplier* of the group. For a group of Lie type,

Group	d
$A_n(q)$	$(n+1, q-1)$
${}^2A_n(q)$	$(n+1, q+1)$
$B_n(q)$	$(2, q-1)$
${}^2B_2(q)$	1
$C_n(q)$	$(2, q-1)$
$D_n(q)$	$(4, q^n-1)$
${}^2D_n(q)$	$(4, q^n+1)$
${}^3D_4(q)$	1
$G_2(q)$	1
${}^2G_2(q)$	1
$F_4(q)$	1
${}^2F_4(q)$	1
$E_6(q)$	$(3, q-1)$
${}^2E_6(q)$	$(3, q+1)$
$E_7(q)$	$(2, q-1)$
$E_8(q)$	1

Table 2.1: Orders of Diagonal Multipliers d of Groups of Lie Type

the Schur multiplier is the direct product of a "diagonal" component and an "exceptional" component, where the order of the exceptional component is a power of the characteristic prime. The order d of the diagonal component appears in the order formulas for the simple groups of Lie type in Table 2.2, and so we document these first in Table 2.1. Our source is [CCN$^+$85, p. xvi, Table 6].

The structure of the cross-characteristic Sylow subgroups of the groups of Lie type is documented in [FF09, Proposition 2.4], which cites [GL83, Section 10] and [GLS98, Section 4.10]. We include the version from [FF09]. Recall that a p-group is said to be *homocyclic* if it is the direct product of isomorphic cyclic subgroups.

G	$d \cdot \lvert G \rvert$
$A_n(q)$	$q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1} - 1)$
${}^2A_n(q)$	$q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1} - (-1)^{i+1})$
$B_n(q)$	$q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$
$C_n(q)$	$q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$
$D_n(q)$	$q^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$
${}^2D_n(q)$	$q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$
${}^3D_4(q)$	$q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$
$G_2(q)$	$q^6(q^6 - 1)(q^2 - 1)$
${}^2G_2(q)$	$q^3(q^3 + 1)(q - 1)$
$F_4(q)$	$q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$
${}^2F_4(q)$	$q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$
$E_6(q)$	$q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1)$
${}^2E_6(q)$	$q^{36}(q^{12} - 1)(q^9 + 1)(q^8 - 1)(q^6 - 1)(q^5 + 1)(q^2 - 1)$
$E_7(q)$	$q^{63}(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$
$E_8(q)$	$q^{120}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^8 - 1)(q^2 - 1)$

Table 2.2: Orders of the Finite Simple Groups of Lie Type

G	$\lvert G\rvert$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Mc	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

Table 2.3: Orders of the Sporadic Finite Simple Groups

Proposition 2.4. *Let $\mathcal{L}$ be a universal group of Lie type over the finite field $\mathbb{F}_q$. There is a finite set $\mathcal{O}$ (depending on $\mathcal{L}$) of positive integers, and "multiplicities" r_m for each $m \in \mathcal{O}$, such that*

$$|\mathcal{L}| = q^\alpha \prod_{m \in \mathcal{O}} (\Phi_m(q))^{r_m}$$

where $\alpha \in \mathbb{N}$ and $\Phi_m(x)$ is the cyclotomic polynomial for the m^{th} roots of unity.

Let p be an odd prime not dividing q and let S be a nontrivial Sylow p-subgroup of $\mathcal{L}$. Let m_0 be the smallest element of $\mathcal{O}$ such that $p \mid \Phi_{m_0}(q)$. Let

$$\mathcal{W} = \{m \in \mathcal{O} \mid m = p^a m_0, a \geq 1\} \quad \text{and} \quad b = \sum_{m \in \mathcal{W}} r_m \tag{2.1}$$

where $b = 0$ if $\mathcal{W} = \varnothing$. Then all of the following hold:

1. *m_0 is the multiplicative order of $q \pmod p$.*
2. *Except in the case where $\mathcal{L} = {}^3D_4(q)$ with $p = 3$, S has a nontrivial normal homocyclic subgroup, S_T, of rank r_{m_0} and exponent $|\Phi_{m_0}(q)|_p$.*
3. *With the same exception as in (2), S is a split extension of S_T by a (possibly trivial) subgroup S_W of order p^b (where b is defined in (2.1)), and S_W is isomorphic to a subgroup of the Weyl group W of the untwisted group corresponding to $\mathcal{L}$. In particular, if $p \nmid |W|$ or if $pm_0 \nmid m$ for all $m \in \mathcal{O}$, then $S = S_T$ is homocyclic abelian.*
4. *If $\mathcal{L} = {}^3D_4(q)$ with $p = 3$ and $|q^2-1|_3 = 3^a$, then S is a split extension of an abelian group of type $(3^{a+1}, 3^a)$ by a group of order 3, and S has rank 2.*
5. *If $\mathcal{L}$ is a classical group (linear, unitary, symplectic, or orthogonal) then every element of order p is conjugate to some element of S_T.*
6. *Except in ${}^3D_4(q)$ (where S_W is not defined), S_W acts faithfully on S_T; and in the simple group $\mathcal{L}/Z(\mathcal{L}) = \overline{\mathcal{L}}$ we have $\overline{S_W} \cong S_W$ acts faithfully on $\overline{S_T}$ except when $p = 3$ with $\mathcal{L} \cong SL_3(q)$ (with $3 \mid q-1$ but $9 \nmid q-1$) or $SU_3(q)$ (with $3 \mid q+1$ but $9 \nmid q+1$).*

7. *If a Sylow p-subgroup of the simple group $\mathcal{L}/Z(\mathcal{L})$ is abelian but not elementary abelian then p does not divide the order of the Schur multiplier of $\mathcal{L}$.*

2.1.2 Algebraic Groups

In addition to their Lie-theoretic derivation, the groups of Lie type appear as subgroups of (infinite) connected algebraic groups (defined over the algebraic closure of the prime subfield). In this context the finite groups arise by Lang's Theorem as the fixed points of Steinberg endomorphisms (which, for simple algebraic groups, are equivalent to Frobenius endomorphisms — cf. [GLS98, Theorem 2.1.1]). For an algebraic group $\mathcal{G}$ and Steinberg endomorphism σ, we often write $G = \mathcal{G}^\sigma = \{g \in \mathcal{G} \mid \sigma(g) = g\}$ for the corresponding finite group.

A *Borel* subgroup of an algebraic group is a maximal connected solvable subgroup, and a closed subgroup containing a Borel subgroup is called *parabolic*. Calling an element whose order is prime to the characteristic of the group *semisimple*, a *torus* is a connected abelian subgroup whose elements are all semisimple. An element whose order is a power of the characteristic prime is said to be *unipotent*, and a unipotent subgroup is one that is closed and contains only unipotent elements. We call a subgroup of an algebraic group *σ-stable* if the Steinberg endomorphism σ maps the subgroup to itself, and we define Borel, parabolic, toral, and unipotent subgroups of the finite group $G = \mathcal{G}^\sigma$ as the fixed points of σ-stable Borel, parabolic, toral, and unipotent subgroups of $\mathcal{G}$, respectively.

If $\mathcal{B}$ is a Borel subgroup of an algebraic group $\mathcal{G}$ in characteristic r, then $\mathcal{B}$ can be written as $\mathcal{B} = \mathcal{U} \rtimes \mathcal{T}$ for some unipotent subgroup $\mathcal{U}$ and torus $\mathcal{T}$. More generally, if $\mathcal{Y}$ is a parabolic subgroup of $\mathcal{G}$, then $\mathcal{Y}$ has a *Levi decomposition* as $\mathcal{Y} = \mathcal{U} \rtimes \mathcal{L}\mathcal{T}$ where $\mathcal{U} = O_r(\mathcal{Y})$ is the *unipotent radical* — the largest normal unipotent subgroup, $\mathcal{T}$ is a torus, and $\mathcal{L}$ is the *Levi factor*. The Levi factor is a central product of groups of Lie type whose structure can be seen as arising from subgraphs of the Dynkin diagram (obtained by removing one or more nodes and the affected edges). With small order exceptions, the Levi factor of the finite group $G = \mathcal{G}^\sigma$ is a central product of quasisimple groups.

All of the preceding material is basic and well known, and can be found in any standard text on algebraic groups, such as [Hum75].

If $\mathcal{T}$ is a maximal torus of the algebraic group $\mathcal{G}$, then the quotient $\mathcal{W} = N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$ is called the *Weyl group* of $\mathcal{G}$. If $G = \mathcal{G}^{\sigma}$ is a finite group of Lie type, then in fact $G = BNB$ for a Borel subgroup B and some subgroup N — this is the *Bruhat decomposition* of G, and the subgroups B and N are called a (B, N)-*pair* for G. The Weyl group W of G is then defined to be $W = N/(B \cap N)$, and occurs as a section of G that is isomorphic to $\mathcal{W}$ when G is untwisted, and to a subgroup of $\mathcal{W}$ otherwise. The Weyl group is generated by involutions that act as reflections on the Euclidean space spanned by the roots of the corresponding Lie algebra. The minimal number of such involutions generating W is the BN-*rank* of the group G. See [GLS94, Section 11] for a more complete description of these concepts.

Table 2.4 lists the structure and order of the Weyl groups of the finite groups of Lie type, which are documented in [Car72, Section 13.3] and [Hum72, Section 12].

We record the structure of the Levi factor of an end node maximal parabolic subgroup (one obtained by removing a single node from the end of the Dynkin diagram) of each of the odd characteristic groups of Lie type with BN-rank at least 2. Note that in general the choice of such a parabolic is not unique, and in particular two such subgroups need not be isomorphic. For our purposes it suffices to identify the structure of one, which is the content of Table 2.5. The information is adapted from [FF09, Table 3B].

We record as well the following important property of parabolic subgroups (distilled from [GLS98, Theorem 2.6.5]).

Theorem 2.5. *Let G be a finite group of Lie type in characteristic r with Y a parabolic subgroup of G, and write $Y = U \rtimes LT$ where $U = O_r(Y)$. Then $C_G(U) = Z(U)Z(G)$.*

Theorem 2.6 describes the structure of the centralizer of a semisimple element of a universal group of Lie type, and is derived from [GLS98, Theorem 4.2.2] (and its proof).

Theorem 2.6. *Let $\widehat{G}$ be a (finite) universal group of Lie type in characteristic r arising as the set of fixed points of an (infinite) simply connected universal algebraic group $\mathcal{G}$ under a Steinberg endomorphism σ, and let $s \in \widehat{G}$ be semisimple. Then:*

(i) $C_{\mathcal{G}}(s) = \mathcal{L}\mathcal{T}$ where $\mathcal{L}$ is a semisimple algebraic subgroup in characteristic r and $\mathcal{T}$ is a maximal torus in $\mathcal{G}$,

Group	Weyl Group	Weyl Group Order
$A_l(q)$	S_{l+1}	$(l+1)!$
$^2A_l(q),\ l$ odd	$Z_2 \wr S_{(l+1)/2}$	$2^{(l+1)/2} \cdot ((l+1)/2)!$
$^2A_l(q),\ l$ even	$Z_2 \wr S_{l/2}$	$2^{l/2} \cdot (l/2)!$
$B_l(q)$	$Z_2 \wr S_l$	$2^l \cdot l!$
$^2B_2(q)$	Z_2	2
$C_l(q)$	$Z_2 \wr S_l$	$2^l \cdot l!$
$D_l(q)$	$E_{2^{l-1}} \rtimes S_l$	$2^{l-1} \cdot l!$
$^2D_l(q)$	$Z_2 \wr S_{l-1}$	$2^{l-1} \cdot (l-1)!$
$^3D_4(q)$	D_{12}	$2^2 \cdot 3$
$G_2(q)$	D_{12}	$2^2 \cdot 3$
$^2G_2(q)$	Z_2	2
$F_4(q)$	$2^{1+4} : (S_3 \times S_3)$	$2^7 \cdot 3^2$
$^2F_4(q)$	D_{16}	2^4
$E_6(q)$	$U_4(2) \cdot 2$	$2^7 \cdot 3^4 \cdot 5$
$^2E_6(q)$	$2^{1+4} : (S_3 \times S_3)$	$2^7 \cdot 3^2$
$E_7(q)$	$Z_2 \times S_6(2)$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
$E_8(q)$	$2 \cdot O_8^+(2) \cdot 2$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

Table 2.4: Weyl Groups of the Groups of Lie Type

$G/Z(G)$	$L/Z(L)$
$L_k(q), k \geq 3$	$L_{k-1}(q)$
$O_k^{\pm}(q), k \geq 7$	$O_{k-2}^{\pm}(q)$
$S_{2k}(q), k \geq 2$	$S_{2k-2}(q)$
$U_k(q), k \geq 4$	$U_{k-2}(q)$
$E_6(q)$	$L_6(q)$
$E_7(q)$	$E_6(q)$
$E_8(q)$	$E_7(q)$
${}^2E_6(q)$	$U_6(q)$
$G_2(q), q > 3$	$L_2(q)$
$F_4(q)$	$S_6(q)$
${}^3D_4(q)$	$L_2(q^3)$

Table 2.5: Levi factors L of groups of Lie type
in odd characteristic with BN-rank at least 2

(ii) $C_{\widehat{G}}(s) = LT$ *where* L *is a central product of groups of Lie type in characteristic* r *and* T *is a torus, and*

(iii) if $\mathcal{L}$ *is nontrivial then so is* L.

We require the following result from [Sei82, (2.7)]:

Lemma 2.7. (Seitz) *Let* $\mathcal{G}$ *be an algebraic group in characteristic* r *with* G *a finite subgroup arising as the set of fixed points of a Steinberg endomorphism. Suppose a subgroup* A *of* G *is an elementary abelian* p*-group of rank at least 2 for an odd prime* $p \neq r$*, and suppose further that* A *is contained in a maximal torus of* $\mathcal{G}$*. Then* A *intersects a proper parabolic subgroup of* G *nontrivially.*

The following standard result in the theory of algebraic groups can be found in [SS70, 5.1]:

Theorem 2.8. *Let* $\widehat{\mathcal{G}}$ *be a universal algebraic group. Then any two commuting semisimple elements of* $\widehat{\mathcal{G}}$ *are contained in a maximal torus.*

2.1.3 General Group Theory

2.1.3.1 Strong Closure

Let P be a subgroup of the finite group G. A subgroup A of P is said to be *strongly closed* in P with respect to G if whenever $g^{-1}ag \in P$ for some $g \in G$ and $a \in A$, then in fact $g^{-1}ag \in A$. We commonly consider the strong closure of a p-subgroup A in a Sylow p-subgroup P; for this case we show in Lemma 2.23 that strong closure is an inherent property of the subgroup A, independent of the choice of Sylow subgroup in which A is contained. Hence it suffices to say that a p-subgroup A is strongly closed with respect to G, or simply that A is strongly closed.

The observation that $\ker \theta|_P$ is strongly closed (see Lemma 3.5) allows us to take advantage of the classification of strongly closed subgroups of finite groups by Flores and Foote. The following simplified version of the main theorem [FF09, Theorem 1.2] suffices for our needs:

Theorem 2.9. (Flores - Foote) *Let p be an odd prime and let G be a finite quasisimple group that possesses a nontrivial strongly closed p-subgroup P_0. If $P_0 \cap Z(G) = 1$ and P_0 is properly contained in the Sylow p-subgroup P of G, then p does not divide the order of $Z(G)$ and, identifying P_0 and P with their isomorphic images in $\overline{G} = G/Z(G)$, $\overline{G}$ belongs to one of the following families:*

(i) $\overline{G}$ is a group of Lie type in characteristic not equal to p and P is homocyclic abelian of the same rank as P_0 but larger exponent.

(ii) $\overline{G} \cong U_3(p^n)$ or $Re(3^n)$ is a group of BN-rank 1; in the unitary case $P_0 \cong E_{p^n}$ is the center of P, and in the Ree group case $p = 3$, n is odd and greater than or equal to 2, and P_0 is either the center or the commutator subgroup of P.

(iii) $\overline{G} \cong G_2(q)$ with $(q, 3) = 1$ and $P_0 = Z(P) \cong Z_3$.

(iv) $\overline{G}$ is one of the following sporadic groups, where in each case P_0 has prime order:

$(p = 3)$: J_2,

$(p = 5)$: Co_3, Co_2, HS, Mc,

$(p = 11)$: J_4.

(v) $\overline{G} \cong J_3$, $p = 3$, *and either* $P_0 = Z(P) \cong E_9$ *or* $P_0 = P' = \Omega_1(P) \cong E_{27}$.

Remark. To obtain this version of the theorem from the original, observe that quasisimplicity implies that $G/\mathcal{O}_{P_0}(G)$ must be isomorphic to a single simple factor L with no "decoration" $D \cdot A_F$, and that $\mathcal{O}_{P_0}(G) = Z(G)$. By the definition of $\mathcal{O}_{P_0}(G)$ (the largest normal subgroup of G whose intersection with P_0 is a Sylow p-subgroup of itself), $P_0 \cap \mathcal{O}_{P_0}(G) = P_0 \cap Z(G)$ is a Sylow p-subgroup of the center of G, hence p does not divide the order of $Z(G)$ by the hypothesis $P_0 \cap Z(G) = 1$. In addition, we have used [FF09, Proposition 2.7] to supplement the information pertaining to the strongly closed 3-subgroups of $G_2(q)$ and J_3.

Proposition 2.10 below appears as [FF09, Corollary 2.8].

Proposition 2.10. (Flores - Foote) *Let p be any prime and let G be a finite simple group that possesses a nontrivial strongly closed p-subgroup P_0 that is properly contained in the Sylow p-subgroup P of G. Then one of the following holds:*

(1) $N_G(P) = N_G(P_0)$,

(2) $|P_0| = 3$ *and* $G \cong G_2(q)$ *for some* q *with* $(q, 3) = 1$, *or*

(3) $|P_0| = 3$, $G \cong J_2$, *and* $N_G(P_0) \cong 3PGL_2(9)$.

2.1.3.2 The Generalized Fitting Subgroup

Let G be a finite group. A subgroup H of G is said to be *subnormal* if there exists a chain of subgroups

$$H \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n \trianglelefteq G.$$

The *components* of G are its subnormal quasisimple subgroups, and the *layer* of G, denoted $E(G)$, is the central product of its components.

The *Fitting subgroup* $F(G)$ is defined as the product of all of the nilpotent normal subgroups of G, or equivalently $F(G)$ is the unique maximal nilpotent normal subgroup of G.

Both $E(G)$ and $F(G)$ are characteristic subgroups of G. Their central product is the *generalized Fitting subgroup* $F^*(G)$. The main theorem concerning the generalized Fitting subgroup is ([Asc86, (31.13)]):

Theorem 2.11. $C_G(F^*(G)) \leq Z(F(G))$.

2.1.3.3 Doubly Transitive Permutation Groups

Let G be a permutation group acting on a set Ω. Recall that G is *doubly transitive* (or *2-transitive*) if for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Omega$ with $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, there is some element $g \in G$ such that $g \cdot \alpha_1 = \beta_1$ and $g \cdot \alpha_2 = \beta_2$. (Continuing in this way we may define k-fold transitivity for any natural number k.) Doubly transitive groups have been extensively studied since Burnside's classical result of 1900 [Bur00, pp. 174-177]:

Theorem 2.12. (Burnside) *A transitive permutation group T of prime degree p is either 2-transitive or solvable. In the latter case, T is either cyclic of order p or else a Frobenius group of order dividing $p(p-1)$ with Frobenius kernel of order p.*

Much of the work by various researchers on 2-transitive groups is nicely summarized in [Cam81]. We have:

Proposition 2.13. *A 2-transitive group has a unique minimal normal subgroup, which is elementary abelian or simple.*

The simple groups that can occur as minimal normal subgroups of 2-transitive groups are listed in Table 2.6, which is adapted from [Cam81, page 8].

2.1.4 Specific Groups

We describe in this section the properties of some specific families of finite groups.

We will need the following two theorems on Ree groups in characteristic 3. Theorem 2.14 is distilled from the main theorem in [War66], and Theorem 2.15 is a small part of [GLS98, Theorem 6.5.5]. Here $\Phi(P)$ is the *Frattini subgroup* — the intersection of all of the maximal subgroups of P.

Theorem 2.14. (Ward) *Let $G \cong {}^2G_2(q)$, $q = 3^n$, n odd and at least 3, and let P be a Sylow 3-subgroup of G. Then $|P| = q^3$, $Z(P)$ is elementary abelian of order q, $P' = \Phi(P)$ is elementary abelian of order q^2 and contains $Z(P)$, and if $x \in P$ has order 9, then $x^3 \in Z(P)$.*

Group	Degree	Transitivity	Notes
A_n	n	n	$n \geq 5$
$L_2(q)$	$(q^d - 1)/(q - 1)$	3	$q \neq 2, 3$
$L_d(q)$	$(q^d - 1)/(q - 1)$	2	$d \geq 2$
$U_3(q)$	$q^3 + 1$	2	$q > 2$
${}^2B_2(q)$	$q^2 + 1$	2	$q = 2^{2a+1}, a \geq 1$
${}^2G_2(q)$	$q^3 + 1$	2	$q = 3^{2a+1}, a \geq 1$
$PSp_{2d}(2)$	$2^{2d-1} \pm 2^{d-1}$	2	$d > 2$
$L_2(11)$	11	2	
$L_2(8)$	28	2	
A_7	15	2	
M_{11}	11	4	
M_{11}	12	3	
M_{12}	12	5	
M_{22}	22	3	
M_{23}	23	4	
M_{24}	24	5	
HS	176	2	
Co_3	276	2	

Table 2.6: 2-Transitive Simple Subgroups of S_n

Remark. The original statement of the theorem specifies $x^3 \in Z(P)$ for all $x \in P - P'$. Since $P' = \Phi(P)$, P/P' is elementary abelian as is P', hence $|x| = 9$ if and only if $x \in P - P'$.

Theorem 2.15. *Let $G \cong {}^2G_2(q)$, $q = 3^n$, n odd and at least 3. Then G has a proper subgroup isomorphic to ${}^2G_2(3) \cong L_2(8).3$.*

Tables 2.7 and 2.8 give the complex character tables for $G \cong SL_2(q)$ where q is a power of an even and odd prime, respectively (when q is even, $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ implies $L_2(q) = SL_2(q)$, so in this case Table 2.7 is the character table for $L_2(q)$ as well). The heading of each column specifies a representative of one of the distinct conjugacy classes followed by the size of the class, and all other notation is as specified following the table. Our source is [Dor71, Chapter 38].

Table 2.7: Characters of $SL_2(q) = L_2(q)$, q even

class:	1	a^l	b^m	c
size:	1	$q(q+1)$	$q(q-1)$	q^2-1
1_G	1	1	1	1
ψ_i	$q-1$	0	$-(\zeta^{im}+\zeta^{-im})$	-1
σ	q	1	-1	0
χ_j	$q+1$	$\rho^{jl}+\rho^{-jl}$	0	1

$$\rho^{q-1} = \zeta^{q+1} = 1, \text{ primitive roots of unity in } \mathbb{C},$$

$$b \text{ is an element of order } q+1 \text{ in } G, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

$$\lambda \text{ is a generator of the multiplicative group } \mathbb{F}_q^\times \text{ of order } q-1,$$

$$1 \le i \le q/2, \quad 1 \le j \le (q-2)/2, \quad 1 \le l \le (q-2)/2, \quad 1 \le m \le q/2.$$

Table 2.8: Characters of $SL_2(q)$, q odd

class:	1	z	a^l	b^m	c	d	zc	zd
size:	1	1	$q(q+1)$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$
1_G	1	1	1	1	1	1	1	1
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\epsilon(q-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}\left(-1+\sqrt{\epsilon q}\right)$	$\frac{1}{2}\left(-1-\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1-\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1+\sqrt{\epsilon q}\right)$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\epsilon(q-1)$	0	$(-1)^{m+1}$	$\frac{1}{2}\left(-1-\sqrt{\epsilon q}\right)$	$\frac{1}{2}\left(-1+\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1+\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1-\sqrt{\epsilon q}\right)$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}\epsilon(q+1)$	$(-1)^l$	0	$\frac{1}{2}\left(1+\sqrt{\epsilon q}\right)$	$\frac{1}{2}\left(1-\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1+\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1-\sqrt{\epsilon q}\right)$
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}\epsilon(q+1)$	$(-1)^l$	0	$\frac{1}{2}\left(1-\sqrt{\epsilon q}\right)$	$\frac{1}{2}\left(1+\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1-\sqrt{\epsilon q}\right)$	$\frac{1}{2}\epsilon\left(1+\sqrt{\epsilon q}\right)$
ψ_i	$q-1$	$(-1)^i(q-1)$	0	$-(\zeta^{im}+\zeta^{-im})$	-1	-1	$(-1)^{i+1}$	$(-1)^{i+1}$
σ	q	q	1	-1	0	0	0	0
χ_j	$q+1$	$(-1)^j(q+1)$	$\rho^{jl}+\rho^{-jl}$	0	1	1	$(-1)^j$	$(-1)^j$

$$\epsilon = (-1)^{(q-1)/2},$$

$\rho \in \mathbb{C}$ is a primitive $(q-1)^{\text{th}}$ root of unity, $\zeta \in \mathbb{C}$ is a primitive $(q+1)^{\text{th}}$ root of unity,

b is an element of order $q+1$ in G,

$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

λ is a generator of the multiplicative group $\mathbb{F}_q^\times$ of order $q-1$,

$$1 \le i \le (q-1)/2, \quad 1 \le j \le (q-3)/2, \quad 1 \le m \le (q-1)/2, \quad 1 \le l \le (q-3)/2.$$

The maximal subgroups of $L_2(q)$ are well known, and are described in [GLS98, Theorem 6.5.1] (attributed to Dickson, Burnside, Moore, and Wiman). We reprise that result here.

Theorem 2.16. *Let $G \cong L_2(q)$, where $q = r^a \geq 5$ and r is a prime. Then G has subgroups of the following isomorphism types (in the indicated cases), and every subgroup of G is isomorphic to a subgroup of one of the following groups:*

(a) Borel subgroups of G, which are Frobenius groups of order $q(q-1)/2$ in odd characteristic and $q(q-1)$ in characteristic 2;

(b) Dihedral groups of order $q-1$ and $q+1$ in odd characteristic and $2(q-1)$ and $2(q+1)$ in characteristic 2;

(c) The groups $PGL_2(r^b)$ if $2b$ divides a, and $L_2(r^b)$ if b is a proper divisor of a;

(d) The alternating group A_5 if 5 divides $|G|$;

(e) The symmetric group S_4 if 8 divides $|G|$; and

(f) The alternating group A_4.

Kleidman and Liebeck [KL90] have described in great detail the subgroup structure of the classical groups of Lie type, extending work of Aschbacher [Asc84]. We include here a very small part of their work (see [KL90, pages 70-72]).

Theorem 2.17. *Let G be a finite classical group.*

(i) If $G \cong SL_n(q)$ for $2 \leq n = ms$ with s prime, then G has a subgroup isomorphic to $GL_m(q^s)$.

(ii) If $G \cong Sp_{2n}(q)$ for $2 \leq n = ms$ with s prime, then G has a subgroup isomorphic to $Sp_{2m}(q^s)$.

(iii) If $G \cong SU_n(q)$ for $3 \leq n = ms$ with $s \neq 2$ prime, then G has a subgroup isomorphic to $GU_m(q^s)$. If moreover n is even, then G has a subgroup isomorphic to $GL_{n/2}(q^2)$.

2.1.5 Character Theory

We consider next some character theoretic results, recalling first some standard background material.

If $H \trianglelefteq G$ and φ is a character of H, the *conjugates* of φ are defined as $\varphi^g(h) = \varphi(ghg^{-1})$ for all $g \in G$, $h \in H$. In this way G acts on the characters of H. The stabilizer of a character φ of H under this action is the subgroup

$$\mathcal{I}_G(\varphi) = \{g \in G \mid \varphi^g = \varphi\},$$

the *inertia group* of φ in G.

The next three results (Lemmas 2.18, 2.19, and 2.20) all derive from *Clifford's Theorem*, which states that the restriction of an irreducible character χ of G to a normal subgroup H is a fixed constant times the sum of the distinct conjugates of an irreducible constituent of χ in H (cf. [Isa76, Theorem 6.2]).

Lemmas 2.18 and 2.19 can be found in [Fei67, pp. 53-55] (a cohomological proof of the second result is given in [Isa76, (11.22)]). Lemma 2.20 is taken from [Gor80, p. 74].

Lemma 2.18. *Let $H \trianglelefteq G$, and suppose that G/H is cyclic. Then any irreducible character of H can be extended to an irreducible character of its inertia group.*

Lemma 2.19. *Let $H \trianglelefteq G$, and let $\chi \in Irr(G)$ with $\varphi \in Irr(H)$ a constituent of $\chi|_H$. Then there exists $\psi \in Irr(\mathcal{I}_G(\varphi))$ such that $\chi = \psi^*$ and $\psi|_H$ is a multiple of φ. If in addition G/H is cyclic, then $\psi|_H = \varphi$.*

Lemma 2.20. *Let G be a nonabelian group of order pq for distinct primes p and q with $p < q$. Suppose that $G \leq GL(V)$ where V is a vector space over a field of characteristic prime to both p and q. Then each element of order p fixes a nonzero vector in V.*

We will use Lemma 2.20 for a group G acting on a vector space over a finite field whose characteristic does not divide the order of G.

Call a subgroup E of a group G *elementary* if it is the direct product of a cyclic group and a p-group for some prime p. Richard Brauer proved the following important theorem (see [Isa76, Chapter 8]):

Theorem 2.21. (Brauer)

(a) *(The "Characterization of Characters":) A class function θ of G is a virtual character if and only if $\theta|_E$ is a virtual character for every elementary subgroup E of G.*

(b) *(The "Theorem on Induced Characters":) Every irreducible character of G is a $\mathbb{Z}$-linear combination of characters induced from elementary subgroups of G.*

2.1.6 Number Theory

Finally, we record a simplified version of Zsigmondy's Theorem [Zsi92], which we will have occasion to cite in Section 5.1.

Theorem 2.22. (Zsigmondy) *Let $q, n \in \mathbb{N}$ with $q \neq 1$. Then except in the cases*

(i) $n = 6$ and $q = 2$, or

(ii) $n = 2$ and $q = 2^k - 1$ for some $k \in \mathbb{N}$,

there exists a prime p such that $p \mid q^n - 1$ but for all $m \in \mathbb{N}$ with $m < n$, $p \nmid q^m - 1$.

2.2 Preliminary Lemmas

In this section we establish some group and character theoretic lemmas. We begin with an equivalent definition of strong closure.

Lemma 2.23. *Let $P \in Syl_p(G)$ with $A \leq P$. Then A is strongly closed in P with respect to G if and only if A is strongly closed in $N_G(A)$ with respect to G.*

Proof. Suppose first that A is strongly closed in P with respect to G. Then clearly $A \trianglelefteq P$, hence P is a Sylow p-subgroup of $N_G(A)$. Let $a \in A$, $g \in G$, and suppose $a^g \in N_G(A)$. By Sylow's Theorem $a^g \in P^n$ for some $n \in N_G(A)$. Thus $a^{gn^{-1}} \in P$, hence $a^{gn^{-1}} \in A$ by our assumption of strong closure. It follows that $a^g \in A^n = A$, hence A is strongly closed in $N_G(A)$ with respect to G.

Now suppose that A is strongly closed in $N_G(A)$ with respect to G. Let $P_1 = N_P(A)$. If $P_1 < P$, then $P_1 < N_P(P_1)$ and there exists $x \in N_P(P_1) - P_1$. Then for any $a \in A$, $a^x \in P_1 \leq N_G(A)$, hence $a^x \in A$ by our assumption of strong closure. But then $x \in N_P(A) = P_1$, a contradiction. Hence $P_1 = P$, i.e. $A \trianglelefteq P$, so $P \leq N_G(A)$. It follows immediately that A is strongly closed in P with respect to G, completing the proof. □

Although the correspondence between the characters of the quotient group G/N and the characters of G whose kernels contain N is basic and well known, we include an explicit statement so that we may refer with precision to its exact nature.

Lemma 2.24. *Let $N \trianglelefteq G$. There is a bijective correspondence between the characters ψ of G having N in their kernels and the characters $\widehat{\psi}$ of G/N given by:*

$$\widehat{\psi}(gN) = \psi(g). \tag{2.2}$$

Under this correspondence $\widehat{\psi}$ is irreducible if and only if ψ is irreducible.

Proof. This follows from consideration of the affording representations. □

We prove next a corollary to Lemma 2.19: the cyclic extension of an abelian group is an M-group.

Corollary 2.25. *Let $H \trianglelefteq G$, and suppose that G/H is cyclic and H is abelian. Then G is an M-group.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. By Lemma 2.19, χ is induced from a character ψ that restricts to an irreducible character of H. Since H is abelian, ψ is linear, hence G is an M-group. □

We obtain a second useful corollary to Lemma 2.19:

Corollary 2.26. *If G is quasisimple and $Z(G)$ is cyclic, then G possesses a faithful irreducible character.*

Proof. Since $Z(G)$ is cyclic, there is a faithful irreducible character φ of $Z(G)$. Choose $\chi \in \mathrm{Irr}(G)$ such that φ is a constituent of $\chi|_{Z(G)}$. By Lemma 2.19 there is a character ψ of $\mathcal{I}_G(\varphi)$ such that $\psi|_{Z(G)} = m\varphi$ for some $m > 0$ and $\psi^* = \chi$. Since G is quasisimple, χ is unfaithful if and only if it is unfaithful on $Z(G)$. But χ is a constant multiple of φ on $Z(G)$:

$$\chi|_{Z(G)}(z) = |G : \mathcal{I}_G(\varphi)|\, m\varphi(z), \tag{2.3}$$

hence χ is faithful. □

We will need the following properties of three dimensional unitary groups in odd characteristic. Although these results are "well-known," they are usually cited without proof in the literature. A good reference for the background material is [Wil09, Section 3.6].

Lemma 2.27. *Let $G \cong SU_3(q)$ or $U_3(q)$ for $q = p^n$ a power of an odd prime, and let P be a Sylow p-subgroup of G. Then:*

(i) A maximal torus H in the Borel subgroup $N_G(P) = P \rtimes H$ acts transitively on the nontrivial elements of $Z(P)$. H is cyclic of order $q^2 - 1$ or $(q^2 - 1)/3$.

(ii) If $t \in G$ is an involution, then $C_G(t)$ contains a subgroup $K \cong SL_2(q)$ of p' index.

Proof. We prove (i) first. Assume that $G \cong SU_3(q)$, so by definition G is the subgroup of $SL_3(q^2)$ preserving a nonsingular conjugate-symmetric sesquilinear form, where conjugation is defined as $\overline{x} = x^q$ for any $x \in \mathbb{F}_{q^2}$. We denote the form by $(\cdot, \cdot)$.

Let V be a nonsingular 3-dimensional unitary space, and write V as the orthogonal direct sum of a hyperbolic plane and an isotropic 1-space. We may specify a hyperbolic basis $\{\mathbf{e}, \mathbf{f}\}$ of the 2-space and then choose a vector $\mathbf{w}$ whose span is the 1-space, so that

$$(\mathbf{e}, \mathbf{e}) = (\mathbf{f}, \mathbf{f}) = (\mathbf{e}, \mathbf{w}) = (\mathbf{f}, \mathbf{w}) = 0 \quad \text{and} \quad (\mathbf{e}, \mathbf{f}) = (\mathbf{w}, \mathbf{w}) = 1. \tag{2.4}$$

With respect to the ordered basis $\{\mathbf{e}, \mathbf{w}, \mathbf{f}\}$, a Sylow p-subgroup P of G is contained in the upper unitriangular matrices, the Borel subgroup $N_G(P)$ is contained in the upper triangular matrices, and in particular the maximal torus H is contained in the diagonal matrices.

We consider first the order of H. Let $h = \mathrm{diag}(a_1, a_2, a_3) \in H$ with $a_1, a_2, a_3 \in \mathbb{F}_{q^2}^{\times}$. Then $a_1 a_2 a_3 = 1$ (since this product is the determinant of h), and h preserves the given form — i.e. $(\mathbf{v_1}, \mathbf{v_2}) = (h\mathbf{v_1}, h\mathbf{v_2})$ for all vectors $\mathbf{v_1}, \mathbf{v_2} \in V$. It suffices to check that the relations in (2.4) are preserved. The action of h on each basis vector is

$$h\mathbf{e} = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = a_1 \mathbf{e}, \tag{2.5}$$

and similarly $h\mathbf{w} = a_2\mathbf{w}$ and $h\mathbf{f} = a_3\mathbf{f}$. The first 4 relations in (2.4) are preserved trivially, for example

$$(\mathbf{e}, \mathbf{e}) = 0 \quad \text{and} \quad (h\mathbf{e}, h\mathbf{e}) = (a_1\mathbf{e}, a_1\mathbf{e}) = a_1\overline{a_1}(\mathbf{e}, \mathbf{e}) = 0 \tag{2.6}$$

for any choice of a_1. Since $(\mathbf{e}, \mathbf{f}) = 1$,

$$(h\mathbf{e}, h\mathbf{f}) = (a_1\mathbf{e}, a_3\mathbf{f}) = a_1\overline{a_3}(\mathbf{e}, \mathbf{f}) = a_1\overline{a_3}. \tag{2.7}$$

Hence h preserves the form only when $a_1\overline{a_3} = 1$, or $a_3 = \overline{a_1^{-1}}$. The condition on the determinant then forces $a_2 = a_1^{-1}\overline{a_1}$; we check that this preserves the remaining relation, $(\mathbf{w}, \mathbf{w}) = 1$:

$$(h\mathbf{w}, h\mathbf{w}) = (a_2\mathbf{w}, a_2\mathbf{w}) = a_2\overline{a_2}(\mathbf{w}, \mathbf{w}) = a_2\overline{a_2} = a_1^{-1}\overline{a_1}\overline{a_1^{-1}}a_1 = 1. \tag{2.8}$$

It follows that our choice of a_1 completely determines h, and every $h \in H$ can be written as

$$h = \begin{pmatrix} a & & \\ & \overline{a}/a & \\ & & 1/\overline{a} \end{pmatrix}, \quad a \in \mathbb{F}_{q^2}^{\times}. \tag{2.9}$$

Hence $H \cong \mathbb{F}_{q^2}^{\times}$, a cyclic group of order $q^2 - 1$.

If instead $G \cong U_3(q)$, we must divide the order of H by the order of $Z(G) \leq H$, the subgroup of scalar matrices. If h is a scalar matrix in H, then $a = \overline{a}/a = 1/\overline{a}$, hence $a\overline{a} = 1$ and $a^2 = \overline{a}$, and it follows that $a^3 = 1$. In particular every nonidentity scalar matrix has order 3, and since H is cyclic either $|Z(G)| = 3$ and $|H| = (q^2 - 1)/3$, or no such element a exists, $SU_3(q) \cong U_3(q)$ (i.e. $Z(G)$ is trivial), and $|H| = q^2 - 1$.

We proceed to prove that H acts transitively on the nontrivial elements of $Z(P)$. Clearly if this is true for $G \cong SU_3(q)$ then it is true as well for $G \cong U_3(q)$, so assume henceforth that $G \cong SU_3(q)$. Then by direct calculation similar to those culminating in (2.9) we see that

$$P = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_{q^2} \text{ and } \overline{a} + c = b + \overline{b} + c\overline{c} = 0 \right\}, \tag{2.10}$$

and in particular

$$Z(P) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{F}_{q^2} \text{ and } b + \overline{b} = 0 \right\}. \tag{2.11}$$

Note that $b + \overline{b}$ is the trace map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. The entries b in the matrices of $Z(P)$ are the kernel of this map, a 1-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^2}$. Thus $Z(P) \cong \mathbb{F}_q$ is an elementary abelian p-group of order q.

Let $1 \neq z_1, z_2 \in Z(P)$, with

$$z_1 = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad z_2 = \begin{pmatrix} 1 & 0 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.12}$$

and $b_i + \overline{b_i} = 0$ for each $i = 1, 2$. Taking h as in (2.9),

$$h^{-1} z_1 h = \begin{pmatrix} 1 & 0 & a^{-1} b_1 \overline{a^{-1}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.13}$$

Hence we must show that there exists $a \in \mathbb{F}_{q^2}^\times$ such that $a^{-1} b_1 \overline{a^{-1}} = b_2$, or equivalently $a^{-1} \overline{a^{-1}} = b_1^{-1} b_2$. Observe that $a^{-1}\overline{a^{-1}}$ is the norm of a^{-1}, and that the norm $x \to x\overline{x} = x^{1+q}$ is a surjective map from $\mathbb{F}_{q^2}^\times$ onto $\mathbb{F}_q^\times$. Hence it suffices to show that $b_1^{-1} b_2 \in \mathbb{F}_q^\times$, or equivalently that $\overline{b_1^{-1} b_2} = b_1^{-1} b_2$. Since $b_2 + \overline{b_2} = 0$, $\overline{b_2} = -b_2$. Similarly $\overline{b_1} = -b_1$, hence $\overline{b_1^{-1}} = (-b_1)^{-1} = -(b_1^{-1})$. It follows that

$$\overline{b_1^{-1} b_2} = (-b_1^{-1})(-b2) = b_1^{-1} b_2, \tag{2.14}$$

which completes the proof of (i).

For (ii), assume $G \cong SU_3(q)$, and let t_0 be the involution $\mathrm{diag}(-1,-1,1)$ with respect to some fixed orthonormal basis. Observe that t_0 preserves the given form, hence is indeed an element of G. Observe further that t_0 acts as the scalar matrix $\mathrm{diag}(-1,-1)$ on a two dimensional subspace W_0 of V, and as the identity on the complementary one dimensional space. It follows that the centralizer of t_0 in $GU_3(q)$ is a subgroup $GU_2(q) \times GU_1(q)$, and the restriction to determinant 1 in $SU_3(q)$ forces the choice of the $GU_1(q)$ factor. Hence $C_G(t_0) \cong GU_2(q)$. The kernel of the determinant map is a normal subgroup $K \cong SU_2(q) \cong SL_2(q)$ of index $q+1$ in $GU_2(q)$, hence p does not divide $|C_G(t_0) : K|$. After taking the quotient by scalars when $G \cong U_3(q)$, the result in (ii) follows once we have shown that the class of involutions is unique.

To that end, let $t \in G$ be an arbitrary involution. We show first that t stabilizes a 1-dimensional isotropic subspace of V. Observe that t is diagonalizable in $GL_3(q^2)$ (since p is odd), hence is conjugate in $GL_3(q^2)$ to t_0. It follows that t stabilizes (and in fact negates) a 2-dimensional subspace W of V. We argue that W contains an isotropic vector. For otherwise let $\mathbf{r}$ be any nonzero vector in W, and after multiplying by a suitable scalar we may assume that $(\mathbf{r},\mathbf{r}) = 1$. Let $\mathbf{s} \in W$ be any nonzero vector orthogonal to $\mathbf{r}$; we may similarly assume that $(\mathbf{s},\mathbf{s}) = 1$. Thus $\{\mathbf{r},\mathbf{s}\}$ is an orthonormal basis for W. The vector $\mathbf{v} = \alpha\mathbf{r} + \mathbf{s}$ is then isotropic for any $\alpha \in \mathbb{F}_{q^2}^{\times}$ such that $\alpha\overline{\alpha} = -1$ (by our remarks on the norm map, such an α exists). Hence W possesses an isotropic vector $\mathbf{v}$. Since $t\mathbf{v} = -\mathbf{v}$, t stabilizes the 1-dimensional isotropic subspace spanned by $\mathbf{v}$, and is therefore contained in a Borel subgroup of Y of G. Since Y is conjugate to the Borel subgroup $N_G(P) = P \rtimes H$ and $H \cong Z_{q^2-1}$ (by part (i)), a Sylow 2-subgroup of Y is cyclic, hence Y has a unique class of involutions. It follows that t is conjugate to t_0 (i.e. G has a unique class of involutions), which completes the proof. □

Lemma 2.28. *If a Sylow p-subgroup of the quasisimple group G is cyclic, then p does not divide the order of the Schur multiplier of $G/Z(G)$.*

Proof. Let P be a cyclic Sylow p-subgroup of G, and assume to begin that G is simple. To obtain a contradiction, suppose that $\widehat{G}$ is a non-split extension of G by a central subgroup Z of order p. Then a Sylow p-subgroup $\widehat{P}$ of $\widehat{G}$ is abelian since $\widehat{P}/Z \cong P$ is cyclic. By Burnside's N/C Theorem there is some

p'-element h of $\widehat{G}$ acting nontrivially on $\widehat{P}$, so by Fitting's Lemma

$$\widehat{P} = C_{\widehat{P}}(h) \times [\widehat{P}, \langle h \rangle]. \tag{2.15}$$

Clearly $Z \leq C_{\widehat{P}}(h)$. Since $\widehat{P}/Z$ is cyclic, h acts fixed point freely on this quotient, which forces $Z = C_{\widehat{P}}(h)$. Thus $\widehat{P}$ splits over Z by (2.15), and it follows by Gaschütz's Theorem that $\widehat{G}$ splits over Z as well, a contradiction.

If G is quasisimple, then a Sylow p-subgroup $\overline{P}$ of $\overline{G} = G/Z(G)$ is either cyclic or trivial. The result follows by observing that the former case has been explicitly proven and in fact so has the latter — if $P = 1$ in the argument above, then h fixes only the identity in $\widehat{P}/Z \cong 1$, hence acts fixed point freely, and $\widehat{P}$ splits over Z as $\widehat{P} = Z \times 1$. □

Chapter 3

Necessary Conditions for the Existence of Unfaithful Minimal Heilbronn Characters

We prove in this chapter one direction of Theorem 1:

Proposition 3.1. *Suppose G is a finite group with an unfaithful minimal Heilbronn character θ. Then θ restricts to an unfaithful character of some Sylow subgroup of G, and if P is a Sylow p-subgroup of G on which θ is unfaithful, then all of the following hold:*

(i) *p is odd,*

(ii) *G is quasisimple with a cyclic center of order prime to p,*

(iii) *P is cyclic,*

(iv) *$N_G(P)$ is a maximal subgroup of G, and*

(v) *Either $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$, or $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$. (In the latter case $\Omega_1(P)$ is also contained in a Borel subgroup $N_G(Q)$ for some Sylow q-subgroup Q of G.)*

If θ is an unfaithful minimal Heilbronn character of G, then it is easy to see that θ is unfaithful on a Sylow subgroup of G: Let $\theta(x) = \theta(1)$ for some nontrivial $x \in G$. Then θ restricts to a character of the proper subgroup $\langle x \rangle$ (G is not cyclic by Lemma 1.4). Since x is in the kernel of $\theta|_{\langle x \rangle}$, it follows that $\theta(x^n) = \theta(1)$ for all integers n, and in particular θ is unfaithful on a Sylow p-subgroup of G for every prime p dividing the order of x. We have proven:

Lemma 3.2. *A finite group G has an unfaithful minimal Heilbronn character if and only if G has a minimal Heilbronn character that restricts to an unfaithful character of a Sylow subgroup.*

We maintain the following assumptions and notation for the duration of Chapter 3:

- θ is a minimal Heilbronn character of the finite group G,
- p is a fixed prime dividing the order of G,
- P is a Sylow p-subgroup of G, and
- $1 \neq P_0 = \ker \theta|_P$.

That p is odd we have already seen — this is Theorem 2.1. We prove that G is quasisimple with a cyclic center of order prime to p in Section 3.2. Using these results and the Classification of the Finite Simple Groups we show that P is cyclic in Section 3.3. We complete the proof of Proposition 3.1 in Section 3.4 by proving that $N_G(P)$ is maximal and, with one family of exceptions, is the unique maximal subgroup of G containing $\Omega_1(P)$.

3.1 Lemmas for Chapter 3

In this section we establish results that will be used throughout the proof of Proposition 3.1. The observations following Definitions 3 and 4 in Section 1.2 are especially germane here. In particular, if H is a proper subgroup of G, then $\ker \theta|_H$ is a normal subgroup of H, and is all of H if and only if $\theta(h) = \theta(1)$ for all $h \in H$.

Lemma 3.3. *If H is a proper subgroup of G generated by conjugates of elements of P_0, then $\ker \theta|_H = H$.*

Proof. Set $H_0 = \ker\theta|_H$, so H_0 is a subgroup of H. Since θ is a class function on G, if $h \in H$ is conjugate to an element of P_0, then $h \in H_0$. Hence the generators of H are all contained in H_0, which implies the result. □

Corollary 3.4. *If M is a proper subgroup of G containing P_0 and $H \trianglelefteq M$, then $[P_0, H] \leq \ker\theta|_H$.*

Proof. It is elementary that $[P_0, H] \leq H$, hence $[P_0, H] = \ker\theta|_{[P_0,H]}$ by Lemma 3.3. The result follows. □

The following observation allows us to utilize the classification of strongly closed subgroups of finite groups (Theorem 2.9):

Lemma 3.5. *P_0 is strongly closed.*

Proof. If $P_0 \trianglelefteq G$ the result is trivial, hence we may assume that $N = N_G(P_0) < G$. Thus $K = \ker\theta|_N$ is a normal subgroup of N by Lemma 1.5, and $P_0 = P \cap K$ is a Sylow p-subgroup of K. Moreover P_0 is normal in K (since it is normal in N), hence in fact P_0 is a characteristic subgroup of K. But then if $g^{-1}ag \in N$ for some $g \in G$ and $a \in P_0$, then $g^{-1}ag \in K$ since θ is a class function on G, hence $g^{-1}ag \in P_0$ by the preceding remarks. □

Suppose a proper subgroup H of G has no proper normal subgroup of order divisible by p. If P_0 intersects H nontrivially, then $\ker\theta|_H$ is a normal subgroup of H containing p-elements, hence must be all of H. We obtain a contradiction to Theorem 2.1 if H has even order. We generalize this argument in the following lemma and its corollary.

Lemma 3.6. *Suppose H is a proper subgroup of G, $N \trianglelefteq H$, the order of H/N is even, and $P_0 \cap N < P_0 \cap H$ (so in particular P_0 intersects H nontrivially). Then H/N has a proper normal subgroup whose order is odd and divisible by p.*

Proof. Set $H_0 = \ker\theta|_H$ and let bars denote passage to the quotient $\overline{H} = H/N$; we will show that $\overline{H_0}$ has the required properties. By Lemma 1.5, H_0 is a normal subgroup of H, and clearly $P_0 \cap H \leq H_0$. Thus the hypothesis $P_0 \cap N < P_0 \cap H$ implies p divides the order of $\overline{H_0}$. If $\overline{H_0} = \overline{H}$, then $H_0 N = H$ so $H/N \cong H_0/H_0 \cap N$ and H_0 has even order, contradicting Theorem 2.1. Hence $\overline{H_0}$ is proper in $\overline{H}$, completing the proof. □

The following corollary is now immediate:

Corollary 3.7. *Suppose N is a normal subgroup of G of order prime to p, and let bars denote passage to the quotient $\overline{G} = G/N$. If $\overline{H}$ is a proper subgroup of $\overline{G}$ of even order having no proper normal subgroup whose order is odd and divisible by p, then $\overline{P_0} \cap \overline{H} = \overline{1}$.*

We will most often use Corollary 3.7 with $N = 1$ or $N = Z(G)$.

3.2 G is Quasisimple

In this section we advance the proof of Proposition 3.1 by establishing that G is quasisimple with a cyclic center of p' order.

That the center of G is cyclic follows from basic representation theory:

Lemma 3.8. *$Z(G)$ is cyclic.*

Proof. If Ψ is any irreducible complex matrix representation of G, $\Psi(z)$ is a scalar matrix for every $z \in Z(G)$ by Schur's Lemma. In particular, Ψ is a homomorphism from $Z(G)$ into a finite subgroup of $\mathbb{C}^{\times}$, hence the image of $Z(G)$ under Ψ is cyclic. The result follows from the observation that G possesses a faithful irreducible representation (by Lemma 1.6). □

We turn to the main result of this section:

Proposition 3.9. *G is quasisimple and p does not divide the order of $Z(G)$.*

Proof. Assume the contrary. We begin by showing that

$$\textit{there exists } 1 \neq K \lhd G \textit{ such that } \theta \textit{ is constant on } K. \tag{3.1}$$

To obtain a contradiction, suppose this is not the case. Then any nontrivial normal subgroup generated by conjugates of elements of P_0 must be all of G by Lemma 3.3. In particular we argue that

(i) $\langle {P_0}^G \rangle = G$,

(ii) $P_0 \cap Z(G) = 1$, and

(iii) If N is any proper normal subgroup of G, then $[N, P_0] = 1$.

These statements follow from considering $K = \langle {P_0}^G \rangle$, $K = \langle (P_0 \cap Z(G))^G \rangle$, and $K = \langle [N, P_0]^G \rangle$, respectively. In each case K satisfies the requirements of (3.1). Since we are proving (3.1) by contradiction, (i) – (iii) must hold.

Let N be a proper normal subgroup of G. Then $[N, P_0]^G = 1^G$ implies $[N, {P_0}^G] = 1$, or since N commutes with the generators of G, $N \leq Z(G)$. Thus every proper normal subgroup of G is contained in the center of G. It follows that $G/Z(G)$ is simple, and in fact nonabelian simple since otherwise $G/Z(G)$ is cyclic and G is abelian (contradicting Lemma 1.4). It follows as well that $G' = G$, since otherwise $G' \leq Z(G)$ implies $G/Z(G)$ is abelian and G is nilpotent, a contradiction as before. Thus we have shown that G is quasisimple. Since we are proceeding by contradiction, we must have that p divides $|Z(G)|$, so P_0 is not a Sylow p-subgroup of G by (ii). But then the classification in Theorem 2.9 implies that p does not divide $|Z(G)|$, providing the contradiction required to establish (3.1).

Now let K be a nontrivial proper normal subgroup of G on which θ is constant. Since θ is constant as well on subgroups of K, we may assume that K is a minimal normal subgroup of G. Thus K is either elementary abelian or the direct product of nonabelian simple groups. In the latter case K has even order, contradicting Theorem 2.1. This proves that K is elementary abelian of odd order.

We show next that G/K is cyclic. The argument follows [Foo97, Lemma 6].

Suppose by way of contradiction that G/K is not cyclic. Then for every $x \in G$ the subgroup $K_x = \langle x, K \rangle$ is proper in G, and in particular $\theta|_{K_x}$ is a character. Since $K \leq \ker \theta|_{K_x}$, $\theta|_{K_x}$ is constant on each coset xK. It follows that θ is a class function on G/K since

$$\theta(g^{-1}xgK) = \theta(g^{-1}xg) = \theta(x) = \theta(xK).$$

Hence θ is a $\mathbb{C}$-linear combination of irreducible characters of G/K, which by Lemma 2.24 implies that θ is a $\mathbb{C}$-linear combination of irreducible characters of G having K in their kernels. Since θ is a virtual character of G and the expression of any class function in terms of irreducible characters is unique, θ is in fact a $\mathbb{Z}$-linear combination of irreducible characters of G having K in their kernels. But since θ is a minimal Heilbronn character of G, some irreducible constituent must occur with negative multiplicity, and that character is necessarily faithful (cf. Lemma 1.6). This contradicts the conclusion that each irreducible constituent of θ has K in its kernel, hence G/K is cyclic.

Finally, since K is elementary abelian and G/K is cyclic, G is an M-group by Corollary 2.25. This contradiction (of Lemma 1.4) completes the proof. □

We obtain an immediate important corollary:

Corollary 3.10. *If $G/Z(G)$ is not isomorphic to one of the groups listed in the conclusion of Theorem* 2.9 *(the classification of strongly closed subgroups of finite groups), then $P_0 = P$.*

Proof. Proposition 3.9 implies all of the hypotheses of the theorem except $P_0 < P$. □

The results of this section also support the following observation:

Corollary 3.11. *Both P and P_0 are isomorphic to their images in the quotient $G/Z(G)$.*

In particular we may identify P and P_0 with their images in $G/Z(G)$. (We will exercise care when passing to quotients and, similarly, to extensions of G, since the properties of θ that hold in G do not translate to these groups — θ is not generally a virtual character of quotients or extensions of G.)

3.3 P is Cyclic

Using the results of the previous section and the Classification of the Finite Simple Groups, we establish here that:

Proposition 3.12. *P is cyclic.*

Some of the results in this section go beyond that which is needed to prove Proposition 3.1. In particular we find many groups that do not possess unfaithful minimal Heilbronn characters. Some of this additional information will be used to prove Theorem 3 and Theorem 5.1 in Chapter 5.

Henceforth let bars denote passage to the quotient $\overline{G} = G/Z(G)$.

We begin with the alternating groups.

Lemma 3.13. *If $\overline{G}$ is an alternating group A_n, then $n = p$, P is cyclic, and θ restricts to a faithful character of a Sylow q-subgroup for every $q \neq p$.*

Proof. Let $\overline{G}$ be an alternating group (of degree at least 5 by Proposition 3.9), and observe that if $n = p$ then P is indeed cyclic.

To obtain a contradiction suppose $n > p$. Since the alternating groups do not appear in the classification in Theorem 2.9, $P_0 = P$ by Corollary 3.10. By Sylow's Theorem any p-cycle in $\overline{G}$ is conjugate to some element of $\overline{P}$, and since conjugation does not alter the cycle type, $\overline{P_0} = \overline{P}$ contains a p-cycle. Hence $\overline{P_0}$ intersects a subgroup $\overline{H} \cong A_{n-1}$ (the stabilizer of a point in $\overline{G}$) nontrivially. Then either $\overline{H}$ is simple, or $n = 5$, $p = 3$, and $\overline{H} \cong A_4$, which admits no proper normal subgroup of order divisible by 3. In either case $\overline{H}$ has even order, contradicting Corollary 3.7. Thus $n = p$.

Finally, if θ is unfaithful on a Sylow q-subgroup of G, then $n = q$ as well, hence $q = p$. □

We address next the groups of Lie type.

Lemma 3.14. *If $\overline{G}$ is a group of Lie type with BN-rank 1 in characteristic p, then $\overline{G} \cong L_2(p)$ and $P = P_0$ is cyclic of order p.*

Proof. The rank 1 groups of Lie type in odd characteristic are (in their adjoint versions) $L_2(q)$, $U_3(q)$, and the Ree groups ${}^2G_2(3^n)$. We show first that $\overline{G}$ cannot belong to the latter two families, and then that $\overline{G} \cong L_2(q)$ only if $q = p$.

Suppose first that $\overline{G} \cong U_3(q)$, $q = p^n$. By the classification in Theorem 2.9 either $P_0 = P$ or $P_0 = Z(P)$, and in either case the center of P, an elementary abelian p-group of order q, is contained in P_0. We have shown in Lemma 2.27 that a cyclic subgroup H of G (of order $(q^2-1)/d$ where $d = 1$ or 3) acts transitively on the nontrivial elements of $Z(P)$, hence if $1 \neq z \in Z(P)$, then the size of the orbit of z under conjugation by H is $q-1$. It follows that $|H : C_H(z)| = q - 1$, or $|C_H(z)| = (q+1)/d$. Thus $C_H(z)$ contains an involution t, and $z \in C_G(t)$. In particular, P_0 intersects $C_G(t)$ nontrivially. It follows from Lemma 2.27 that P_0 intersects a subgroup $K \cong SL_2(q)$ of $C_G(t)$ nontrivially, contradicting Corollary 3.7. Thus G is not a BN-rank 1 unitary group in characteristic p.

Suppose next that G is a Ree group ${}^2G_2(3^n)$ with n odd and at least 3 (since ${}^2G_2(3) \cong L_2(8).3$ is not quasisimple). We may dispense with bars since the Schur multiplier is trivial (cf. Table 2.1). By the classification in Theorem 2.9, either $P_0 = P$, $P_0 = Z(P)$, or $P_0 = P'$, and in any case $Z(P) \leq P_0$ by Theorem 2.14. G has a proper subgroup $H \cong {}^2G_2(3)$ by

Theorem 2.15, and we may assume that $P \cap H \in \mathrm{Syl}_p(H)$. A parabolic subgroup of H is dihedral of order 18 (cf. [CCN+85, p. 6]), and in particular there is an element $h \in P \cap H$ of order 9. Then $h^3 \in Z(P)$ by Theorem 2.14, so P_0 intersects H nontrivially. This contradicts Corollary 3.7, so G is not ${}^2G_2(3^n)$.

Finally, suppose that $\overline{G} \cong L_2(q)$, $q = p^n$. Then P is elementary abelian (the strictly upper triangular matrices form a copy of the additive group $\mathbb{F}_q$). Also $P_0 = P$ by Corollary 3.10 (since the groups $L_2(q)$ do not appear in the classification of Theorem 2.9). Hence P is cyclic precisely when $q = p$, and we must show that the assumption $q \neq p$ generates a contradiction. When $q \neq p$ the subgroup $\overline{H} \cong L_2(p)$ obtained by restricting the entries of G to to the prime subfield $\mathbb{F}_p$ is proper. Then except when $p = 3$, $\overline{H}$ is simple, intersects $\overline{P_0}$ nontrivially, and has even order, contradicting Corollary 3.7. When $p = 3$, $\overline{H} \cong A_4$ admits no normal subgroup of order divisible by 3, hence the same contradiction applies. Hence if G is a BN-rank 1 linear group over $\mathbb{F}_q$, then $q = p$ and $\overline{G} \cong L_2(p)$. □

Lemma 3.15. *$\overline{G}$ is not a group of Lie type with BN-rank at least 2 in characteristic p.*

Proof. Assume the contrary. It is immediate from Corollary 3.10 and the classification in Theorem 2.9 that $P_0 = P$. Hence if Y is any parabolic subgroup of G, P_0 intersects a conjugate of Y nontrivially. In fact P_0 intersects a conjugate of the Levi factor of Y, since the Levi factor is a group of Lie type in characteristic p. Table 2.5 gives the structure of a Levi factor L of an end node maximal parabolic subgroup for each of the odd characteristic groups of Lie type with BN-rank at least two. Unless $\overline{G}$ is isomorphic to $L_3(3)$, $G_2(3)$, $S_4(3)$, or $U_4(3)$, L is quasisimple and therefore provides the required contradiction of Corollary 3.7. In the excepted cases we obtain contradictions of Corollary 3.7 by considering subgroups $\overline{H} \cong A_4$ of $L_3(3)$, $\overline{H} \cong L_3(3)$ of $G_2(3)$, $\overline{H} \cong A_6$ of $S_4(3)$, and $\overline{H} \cong U_3(3)$ of $U_4(3)$ (cf. [CCN+85]). This completes the proof of the lemma. □

Lemma 3.16. *If $\overline{G}$ is a group of Lie type in characteristic other than p, then P is cyclic.*

Proof. Assume by way of contradiction that $\overline{G}$ is a group of Lie type in characteristic $r \neq p$ and P is not cyclic. We begin by establishing that

$$\textit{there is a subgroup } A_0 \textit{ of } P_0 \textit{ with } A_0 \cong Z_p \times Z_p. \tag{3.2}$$

Observe first that P_0 is not cyclic: If $P_0 = P$ then this is by assumption. Otherwise P is homocyclic of the same rank as P_0 by Theorem 2.9 part (i), so if P_0 is cyclic then so is P, a contradiction. Thus P_0 is a noncyclic odd-order p-group, and (3.2) follows.

We prove next that

$$\textit{θ acts faithfully on every r-subgroup of G.} \tag{3.3}$$

If not, then we may substitute the prime r for p in Lemmas 3.14 and 3.15 to conclude that $\overline{G}$ is isomorphic to $L_2(r)$. This contradicts (3.2) since the cross characteristic odd order Sylow subgroups of $L_2(r)$ are all cyclic. Hence (3.3) holds.

We argue that

$$\textit{p divides the order of the Schur multiplier of G.} \tag{3.4}$$

Let $\widehat{G}$ be the (finite) universal covering group of G, and let $\mathcal{G}$ be an (infinite) simply connected universal algebraic group defined over the algebraic closure of $\mathbb{F}_r$ having $\widehat{G}$ as the set of fixed points of a Steinberg endomorphism. Suppose, to obtain a contradiction, that (3.4) does not hold. Then A_0 remains abelian in $\widehat{G}$ and in $\mathcal{G}$, hence is contained in a maximal torus of $\mathcal{G}$ by Theorem 2.8. By Lemma 2.7 a nontrivial subgroup A of A_0 is contained in a proper parabolic subgroup Y of $\widehat{G}$. Writing $Y = R \rtimes L$ with R the unipotent radical of Y, $[A, R] \leq \ker \theta|_R$ by Corollary 3.4. By Theorem 2.5, $C_{\widehat{G}}(R) = Z(R)Z(\widehat{G})$, so A cannot centralize R, and thus $[A, R]$ is a nontrivial r-subgroup of $\widehat{G}$ on which θ is constant. Observing that (3.3) holds in $\widehat{G}$, this contradiction establishes (3.4). An immediate consequence is

$$P_0 = P \tag{3.5}$$

since otherwise Theorem 2.9 part (i) and Proposition 2.4 part (7) together contradict (3.4).

The orders of the cross-characteristic components of the Schur multipliers of the groups of Lie type are given in Table 2.1. Since p is odd it is clear from the table that either $\overline{G} \cong A_n(q)$ or ${}^2A_n(q)$ with p dividing $(q-1)$ or $(q+1)$, respectively, or $\overline{G} \cong E_6(q)$ or ${}^2E_6(q)$ with $p = 3$. We complete the proof of the lemma by showing that these cases do not occur.

If $\overline{G} \cong E_6(q)$ or ${}^2E_6(q)$, then $p = 3$ divides $|L/Z(L)|$ for the Levi factor L of the end node maximal parabolic subgroup listed in Table 2.5, contradicting Corollary 3.7. Hence $\overline{G}$ is not isomorphic to $E_6(q)$ or ${}^2E_6(q)$.

It remains to consider $\overline{G} \cong A_n(q)$ or ${}^2A_n(q)$, with p dividing $q-1$ or $q+1$, respectively. With respect to a suitable basis the upper triangular matrices in G form a Borel subgroup B, with $B = R \rtimes T$ for R a Sylow r-subgroup appearing as the upper unitriangular matrices and T a torus appearing as the diagonal matrices. The torus T is the direct product of cyclic components of order $q-1$ in the linear case and q^2-1 in the unitary case (the generators being chosen freely from the elements of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$, respectively). Hence p divides the order of T, so $P_0 = P$ intersects T nontrivially. Moreover if $T_0 = P_0 \cap T$, then T_0 acts nontrivially on R by Theorem 2.5, and in particular $1 \neq [T_0, R] \leq R \cap \ker\theta|_B$. But then θ is unfaithful on the r-subgroup $[T_0, R]$, contradicting (3.3). This completes the proof of the lemma. □

To facilitate our discussion of the sporadic groups, we extend the definition of p-singular, usually applied to an *element* of order divisible by p, to subgroups:

Definition 7. *A subgroup H of a group G is said to be **p-singular** if and only if p divides the order of H.*

Lemma 3.17. *If $\overline{G}$ is a sporadic simple group, then P is cyclic and one of the following holds:*

(i) $\overline{G} \cong J_1$ *and* $p = 19$,

(ii) $\overline{G} \cong M_{23}$ *and* $p = 23$,

(iii) $\overline{G} \cong Ly$ *and* $p = 37$ *or* 67,

(iv) $\overline{G} \cong J_4$ *and* $p = 29$ *or* 43,

(v) $\overline{G} \cong Fi_{24}'$ *and* $p = 29$,

(vi) $\overline{G} \cong B$ *and* $p = 47$.

Remark. We include the Tits group ${}^2F_4(2)'$ in our treatment below even though Lemma 3.16 has already established Proposition 3.12 for this group. We obtain the stronger result that in fact $\overline{G}$ is not the Tits group.

Proof. Table 3.1 lists each sporadic simple group $\overline{G}$ together with the odd primes p dividing the order of $\overline{G}$. For each pair $(\overline{G}, p)$, a proper subgroup $\overline{H}$ intersecting P_0 nontrivially and admitting no normal p-singular subgroup is specified, if possible. If such a subgroup $\overline{H}$ exists, then it provides a contradiction of Corollary 3.7 for the given prime p. Before we address the details involved in ensuring that $\overline{H}$ intersects P_0 nontrivially, we remark that such subgroups $\overline{H}$ do indeed exist and are specified in Table 3.1 for all pairs $(\overline{G}, p)$ where a Sylow p-subgroup of $\overline{G}$ is not cyclic, with the single exception of the pair $(J_3, 3)$. This proves the lemma in all cases but this one, which we treat separately. Observe further that the cases where such a subgroup $\overline{H}$ cannot be found (indicated by a "—" in the table) are the only candidates for unfaithful minimal Heilbronn characters among the sporadic simple groups.

The cases in which P_0 may be a proper subgroup of P in a sporadic simple group are classified in Theorem 2.9 and are:

1. $p = 3$, $\overline{G}$ isomorphic to J_2 or J_3;
2. $p = 5$, $\overline{G}$ isomorphic to HS, Mc, Co_3, or Co_2;
3. $p = 11$, $\overline{G}$ isomorphic to J_4.

In these cases, except for $\overline{G} \cong J_3$ which is addressed below, Table 3.1 specifies a subgroup $\overline{H}$ containing a full Sylow p-subgroup of $\overline{G}$, thereby ensuring that $\overline{P_0}$ intersects $\overline{H}$ nontrivially (the same is true for $\overline{G} \cong {}^2F_4(2)'$, which is not technically sporadic, and each odd prime dividing its order). In all other cases it suffices to have p divide the order of $\overline{H}$.

Aside from these considerations, nothing should be inferred from the selection of a particular subgroup $\overline{H}$ for inclusion in the table — there are in general multiple reasonable choices.

Thus, unless $\overline{G} \cong J_3$ and $p = 3$, the lemma is proven by Table 3.1 and the comments above.

Suppose then that $\overline{G}$ is isomorphic to J_3 and $p = 3$. We show first that

$$Z(P) \leq P_0. \tag{3.6}$$

This is clear when $P_0 = P$ or $P_0 = Z(P)$; by Theorem 2.9 the only other possibility is $P_0 = \Omega_1(P) \cong E_{27}$, in which case $Z(P) \cong E_9$ implies $Z(P) \leq \Omega_1(P)$, establishing (3.6).

Choosing $a \in P$ with a in the noncentral conjugacy class of 3-elements,

$$Z(P) \leq N_G(\langle a \rangle) \cong (Z_3 \times A_6) \rtimes Z_2 \tag{3.7}$$

by [CCN+85]. In particular, $|Z(P)| = 9$ implies $Z(P)$ intersects the A_6 component of the normalizer nontrivially, hence so does P_0. This contradicts Corollary 3.7, completing the proof of the lemma. □

By the preceding results and the Classification of the Finite Simple Groups, the proof that P is cyclic (Proposition 3.12) is now complete.

Table 3.1: Some p-singular Subgroups of Sporadic Simple Groups

$\overline{G}$	p	$\overline{H}$
M_{11}	$3, 5, 11$	$L_2(11)$
M_{12}	$3, 5, 11$	M_{11}
J_1	$3, 5, 11$	$L_2(11)$
J_1	7	$2^3 : 7$
J_1	19	—
M_{22}	$3, 5, 7$	A_7
M_{22}	11	$L_2(11)$
J_2	$3, 7$	$U_3(3)$
J_2	5	A_5
M_{23}	$3, 5, 7, 11$	M_{22}
M_{23}	23	—
${}^2F_4(2)'$	$3, 13$	$L_3(3)$
${}^2F_4(2)'$	5	$L_2(25)$
HS	$3, 5, 7$	$U_3(5)$
HS	11	M_{11}
J_3	3	see text
J_3	$5, 19$	$L_2(19)$
J_3	17	$L_2(17)$
M_{24}	$3, 5, 7, 11, 23$	M_{23}
Mc	$3, 5, 7$	$U_3(5)$
Mc	11	M_{11}
He	$3, 5, 17$	$S_4(4)$
He	7	$2^2 \cdot L_3(4)$
Ru	$3, 5, 7, 29$	$L_2(29)$
Ru	13	$L_2(13)$
Suz	$3, 5, 7, 13$	$G_2(4)$
Suz	11	M_{12}
$O'N$	$3, 5, 7, 11, 19$	J_1
$O'N$	31	$L_2(31)$
Co_3	$3, 5, 7, 11$	Mc
Co_3	23	M_{23}
Co_2	$3, 5, 7, 11$	Mc
Co_2	23	M_{23}
Fi_{22}	$3, 5, 7, 11$	$2 \cdot U_6(2)$
Fi_{22}	13	$O_7(3)$
HN	$3, 5, 7, 11$	A_{12}

(continued on next page)

Table 3.1: (continued)

$\overline{G}$	p	$\overline{H}$
HN	19	$U_3(8)$
Ly	$3, 5, 7, 31$	$G_2(5)$
Ly	11	$2 \cdot A_{11}$
Ly	$37, 67$	—
Th	$3, 5, 7, 31$	$2^5 \cdot L_5(2)$
Th	13	${}^3D_4(2)$
Th	19	$U_3(8)$
Fi_{23}	$3, 5, 7, 11, 13$	Fi_{22}
Fi_{23}	17	$L_2(17)$
Fi_{23}	23	M_{23}
Co_1	$3, 5, 7, 11, 23$	Co_2
Co_1	13	$3 \cdot Suz$
J_4	$3, 5, 11, 37$	$U_3(11)$
J_4	$7, 23$	M_{24}
J_4	31	$2^{10} : L_5(2)$
J_4	$29, 43$	—
Fi'_{24}	$3, 5, 7, 11, 13, 17, 23$	Fi_{23}
Fi'_{24}	29	—
B	$3, 5, 7, 11, 13$	Fi_{22}
B	$17, 19$	${}^2E_6(2)$
B	23	$2_+^{1+22} \cdot Co_2$
B	31	$5^3 \cdot L_3(5)$
B	47	—
M	$3, 5, 7, 11, 13, 17, 19, 23, 31, 47$	$2 \cdot B$
M	29	$3 \cdot Fi'_{24}$
M	41	$3^8 \cdot O_8^-(3)$
M	59	$L_2(59)$
M	71	$L_2(71)$

Some p-singular Subgroups of Sporadic Simple Groups

3.4 $N_G(P)$ is Maximal

We establish in this section the final two conclusions of Proposition 3.1: $N_G(P)$ is maximal, and, with one exceptional family, $N_G(P)$ is the unique maximal subgroup containing P_0.

Lemma 3.18. $N_G(P) = N_G(P_0) = N_G(\Omega_1(P))$.

Proof. We show first that $N_G(P) = N_G(P_0)$, which holds trivially if $P_0 = P$. Otherwise $P_0 < P$ and Proposition 2.10 yields the desired result unless $p = 3$ and $\overline{G} \cong J_2$ or $\overline{G} \cong G_2(q)$ with $3 \nmid q$. These cases do not occur — see Table 3.1 for J_2; for $G_2(q)$ observe that a Sylow 3-subgroup is nonabelian — hence, in particular, not cyclic — when $3 \nmid q$ (cf. Theorem 2.9).

Since P is cyclic $\Omega_1(P) \leq P_0$, and in particular $\Omega_1(P)$ is a characteristic subgroup of P_0. It follows that $\Omega_1(P)$ is strongly closed, hence $N_G(P) = N_G(\Omega_1(P))$ by replacing P_0 with $\Omega_1(P)$ in the preceding argument. This completes the proof of the lemma. □

Recall that $E(G)$, $F(G)$, and $F^*(G)$ denote the layer, Fitting subgroup, and generalized Fitting subgroup of G, respectively (see Section 2.1.3.2).

Lemma 3.19. *If M is a proper subgroup of G containing P_0, then P_0 acts trivially on $E(M)$.*

Proof. Let $E = E(M)$. By basic properties of commutators, $[E, P_0] \trianglelefteq \langle E, P_0 \rangle$. Since E is normal in M, $[E, P_0] \leq E$, so in fact $[E, P_0] \trianglelefteq E$. By Lemma 3.3, $[E, P_0] \leq \ker \theta|_E$, so in particular $[E, P_0]$ is an odd-order normal subgroup of E. Since E is the central product of quasisimple groups, it follows that $[E, P_0] \leq Z(E)$. Thus $[E, P_0, E] = [P_0, E, E] = 1$, so $[E, E, P_0] = 1$ by the Three Subgroup Lemma. Since E is perfect, $[E, P_0] = 1$, which was to be shown. □

Lemma 3.20. *$N_G(P)$ is a maximal subgroup of G.*

Proof. Let M be a maximal subgroup of G containing $N_G(P) = N_G(P_0)$; we show that P_0 is normal in M.

We establish first that

$$P_0 \text{ is contained in the commutator subgroup } M' \text{ of } M. \tag{3.8}$$

By the Schur-Zassenhaus Theorem, $N_G(P) = P \rtimes H$ for some p'-subgroup H. By Fitting's Lemma $P = C_P(H) \times [P, H]$, but since P is cyclic one component must be trivial. If $P = C_P(H)$, then P commutes with the generators of $N_G(P)$, so $N_G(P) \leq C_G(P)$, and therefore $N_G(P) = C_G(P)$. But then G has a normal p-complement by Burnside's N/C Theorem, a contradiction since G is quasisimple. Hence $P = [P, H]$, proving (3.8).

We proceed to show that

$$P_0 \textit{ acts trivially on the Fitting subgroup } F = F(M). \tag{3.9}$$

Since F is the direct product of its Sylow subgroups, it suffices to show that P_0 acts trivially on these. To that end, let R be a Sylow subgroup of F, and suppose, to obtain a contradiction, that $[P_0, R] \neq 1$. Since R is characteristic in the characteristic subgroup F, R is normal in M. Thus $1 \neq [P_0, R] \leq \ker\theta|_R$ (by Corollary 3.4), so R is cyclic by Proposition 3.12. Since $C_M(R)$ is the kernel of the action of M on R by conjugation, $M/C_M(R)$ is isomorphic to a subgroup of the abelian group $Aut(R)$. It follows that $M' \leq C_M(R)$, and since $P_0 \leq M'$ by (3.8), $[P_0, R] = 1$, a contradiction. This establishes (3.9).

By Lemma 3.19, P_0 acts trivially on $E = E(M)$ as well, hence P_0 centralizes the generalized Fitting subgroup $F^*(M) = EF$. It follows by Theorem 2.11 that P_0 is contained in the center of the Fitting subgroup. Then since P is cyclic, P_0 is characteristic in a Sylow p-subgroup of F, which is in turn characteristic in F, which is characteristic in M. Hence P_0 is normal in M, completing the proof. □

Lemma 3.21 sharply restricts the groups of Lie type in which θ may be unfaithful on a parabolic subgroup:

Lemma 3.21. *Let G be a group of Lie type in characteristic r and suppose $\ker\theta|_Y$ is nontrivial for some parabolic subgroup Y of G. Then r is odd and $G/Z(G) \cong L_2(r)$.*

Proof. We show that

$$r \textit{ is odd and a Sylow } r\textit{-subgroup of } G \textit{ is cyclic.} \tag{3.10}$$

The result then follows from the observation that the only simple groups of Lie type with cyclic equi-characteristic Sylow subgroups are the groups $L_2(r)$ with r prime.

Let s be a prime dividing the order of $\ker\theta|_Y$ and let S be a Sylow s-subgroup of Y. If $s = r$ then (3.10) follows immediately from Theorem 2.1 and Proposition 3.12. Otherwise S acts nontrivially on the unipotent radical R of Y by Theorem 2.5, hence $1 \neq [S, R] \leq \ker\theta|_R$ (by Corollary 3.4). By Theorem 2.1 and Proposition 3.12 we obtain (3.10), completing the proof. □

We will show that in the general case $N_G(P)$ is the unique maximal subgroup containing P_0. To that end we consider next the action in a group of Lie type of P_0 on the Fitting subgroup of an arbitrary maximal subgroup M containing P_0, i.e. one in which P_0 is not assumed to be normal.

Lemma 3.22. *Suppose G is a group of Lie type in characteristic r and M is a maximal subgroup of G containing P_0. Then either P_0 centralizes the Fitting subgroup $F = F(M)$, or $G/Z(G) \cong L_2(r)$, r is an odd prime, p divides $r - 1$, and M is the normalizer of a Sylow r-subgroup of G.*

Proof. We show that P_0 centralizes each Sylow subgroup of F. Let S be a Sylow s-subgroup of F, and observe that $S \trianglelefteq M$ since S is characteristic in F (so in fact $M = N_G(S)$ by the hypothesis of its maximality). Suppose, to obtain a contradiction, that P_0 acts nontrivially on S. Then since $1 \neq [P_0, S] \leq \ker\theta|_S$ (by Corollary 3.4), s is odd and S is cyclic by Proposition 3.12. If $s = r$ is the characteristic of G, then $G/Z(G)$ is isomorphic to $L_2(r)$ by Lemma 3.21. But then p divides $|Aut(S)| = r-1$, which is precisely the specified exception. Hence $s \neq r$.

Let $\widehat{G}$ be the universal cover of G. By Lemma 2.28, neither s nor p divides the order of the Schur multiplier of G, so we may identify S and P with their isomorphic images in $\widehat{G}$ (taking care, though, to restrict our dependence on the properties of θ in the ensuing argument to the group G, as θ is not generally a virtual character of $\widehat{G}$). We establish first that

$$\textit{neither } S \textit{ nor } P_0 \textit{ intersects a parabolic subgroup of } \widehat{G} \textit{ nontrivially.} \quad (3.11)$$

For otherwise S or P_0 intersects a corresponding parabolic subgroup of G nontrivially, and $G/Z(G) \cong L_2(r)$ by Lemma 3.21. But then the normalizer of S is dihedral by Theorem 2.16, contradicting the assumption that P_0 acts nontrivially on S (since $p \nmid |N_G(S)/C_G(S)| = 2$).

We consider first the case where G is a classical group. Let V denote the underlying natural module, and let n denote the dimension of V. Assuming without loss of generality that $P_0 = \Omega_1(P)$, P_0 fixes some nonzero vector

$\mathbf{v} \in V$ by Lemma 2.20 applied to the subgroup $\Omega_1(S) \rtimes P_0$ of G. If $\mathbf{v}$ is isotropic (which is always the case when G is linear or symplectic), then P_0 intersects a parabolic subgroup $\widehat{Y}$ of $\widehat{G}$ nontrivially (as the parabolic subgroups are precisely the stabilizers of isotropic subspaces), contradicting (3.11). If instead $\mathbf{v}$ is not isotropic (so G is unitary or orthogonal), then observe that P_0 acts on the perpendicular space $\mathbf{v}^\perp$ of dimension $n-1$. It follows that P_0 normalizes the stabilizer $\widehat{H}$ of $\mathbf{v}^\perp$, a subgroup of the same Lie type and characteristic as $\widehat{G}$ but of smaller rank. Let H be the subgroup of G corresponding to $\widehat{H}$ in $\widehat{G}$ (so $P_0 \leq N_G(H)$), and let R be a Sylow r-subgroup of H. If P_0 centralizes R, then $P_0 \leq N_G(R)$, a parabolic subgroup, contradicting (3.11). Hence $1 \neq [P_0, R] \leq \ker\theta|_H \trianglelefteq H$, and in particular $\ker\theta|_H$ maps onto a nontrivial odd-order normal subgroup of $H/Z(H)$ (by the usual arguments, including Theorem 2.1, Corollary 3.4, Proposition 3.12, and Lemma 2.28). But $H/Z(H)$ is simple unless $G/Z(G) \cong U_3(3)$, $U_4(2)$, or $P\Omega_6^{\pm}(2)$, in which case $H/Z(H)$ is isomorphic to $L_2(3) \cong S_4$, $U_3(2) \cong 3^2.Q_8$, or $P\Omega_5(2) \cong S_6$, respectively. The first and last of these groups have no nontrivial odd order normal subgroups, and if $p = 3$ in $G/Z(G) \cong U_4(2)$, then p divides the order of a parabolic. This establishes that G is not classical.

It remains to consider the exceptional (twisted and untwisted) groups of Lie type. We eliminate these by passing to an (infinite) simply connected universal algebraic group $\mathcal{G}$ having $\widehat{G}$ as the set of fixed points of a Steinberg endomorphism. By our identification of S and P with their isomorphic images in $\widehat{G} \leq \mathcal{G}$, we may consider S and P as subgroups of $\mathcal{G}$. (Note that the argument that follows for the untwisted exceptional groups of Lie type works as well for the untwisted classical groups, thereby eliminating these groups redundantly.)

We begin by proving that

$$p \text{ divides the order of the Weyl group } \mathcal{W} \text{ of the algebraic group } \mathcal{G}. \quad (3.12)$$

By Theorem 2.6, $C_{\widehat{G}}(S) = LT$ where L is a central product of groups of Lie type in characteristic r. If L is nontrivial, then S centralizes — and therefore normalizes — some nontrivial r-subgroup of $\widehat{G}$, hence is contained in a parabolic subgroup of $\widehat{G}$, contradicting (3.11). Hence $L = 1$, and it follows from Theorem 2.6 that $C_{\mathcal{G}}(S) = \mathcal{T}$ is a maximal torus. Thus P_0 normalizes but does not centralize $\mathcal{T}$ (since the same is true of the action of P_0 on S), and (3.12) follows from the observation that $\mathcal{W} \cong N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$.

If G is an untwisted group, then the Weyl group W of G is isomorphic to

$\mathcal{W}$. By (3.12) and Lemma 3.6, there is a proper, normal, odd-order subgroup H of W with p dividing the order of H. Referring to Table 2.4 (which lists the isomorphism types and orders of the Weyl groups of the groups of Lie type), this is a contradiction unless $p = 3$ and $W \cong S_3$ or D_{12} (for $G/Z(G) \cong L_3(q)$ or $G_2(q)$, respectively). As $P_0 \cong Z_3$ is contained in a parabolic subgroup in these cases, G is not an untwisted group.

It remains to consider $G/Z(G)$ a Suzuki or Ree group, ${}^3D_4(q)$, or ${}^2E_6(q)$. In the case of the Suzuki groups, $\mathcal{W}$ is a 2-group, contradicting (3.12). In ${}^3D_4(q)$ and the Ree groups, $p = 3$ is the only odd prime dividing the order of $\mathcal{W}$, and $p = 3$ divides the order of a parabolic. If $G/Z(G) \cong {}^2E_6(q)$, then the odd primes dividing the order of $\mathcal{W}$ are $p = 3$ and $p = 5$. As it is clear that $p = 3$ divides the order of a parabolic subgroup, we may assume that $p = 5$. An end node maximal parabolic subgroup of G has a Levi factor L isomorphic to $U_6(q)$ (see Table 2.5). By Table 2.1, $p = 5$ is prime to the Schur multiplier of L, hence by Table 2.2 the order of L is divisible by $q^4 - 1$ (where $q = r^\alpha$ and $p = 5$ are coprime). But $p = 5$ divides $(q^2 + 1)(q^2 - 1)$ for all q coprime to 5 as ± 1 are the only squares modulo 5. Hence $p = 5$ divides the order of a parabolic subgroup in this case as well, completing the proof. □

Proposition 3.23. *Either $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$, or $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q - 1$. (In the latter case $\Omega_1(P)$ is also contained in a Borel subgroup $N_G(Q)$ for some Sylow q-subgroup Q of G.)*

Proof. Let M be a maximal subgroup of G containing $\Omega_1(P)$; we will show in general that $\Omega_1(P)$ is normal in M. Since $N_G(P) = N_G(\Omega_1(P))$ by Lemma 3.18, this will prove the result for all but the excepted case.

Suppose first that G is a group of Lie type, and if $\overline{G} \cong L_2(q)$ with q an odd prime and p dividing $q - 1$, then M is not the normalizer of a Sylow q-subgroup of G. Then by Lemmas 3.19 and 3.22, P_0 acts trivially on the generalized Fitting subgroup $F^*(M)$. It follows, as in the last paragraph of the proof of Lemma 3.20, that P_0 is contained in the center of $F(M)$ and is thereby normal in M. Since $\Omega_1(P)$ is a characteristic subgroup of P_0, $\Omega_1(P)$ is normal in M as well.

If instead $\overline{G} \cong L_2(q)$, q is an odd prime, p divides $q - 1$, and M is the normalizer of a Sylow q-subgroup, then the argument in the preceding paragraph ensures that $\Omega_1(P)$ is contained only in its own normalizer and conjugates of M. Moreover $\Omega_1(P)$ is indeed contained in the normalizer M of a Sylow q-subgroup: $M \cong Z_q \rtimes Z_{q-1}$ or $Z_q \rtimes Z_{(q-1)/2}$ according to whether $G \cong SL_2(q)$ or $L_2(q)$, respectively (cf. Theorem 2.16), and in either case p divides the order of M. Since P is cyclic, $\Omega_1(P)$ is contained in some conjugate of every p-singular subgroup of G, and in particular $\Omega_1(P) \leq M$ for a suitable choice of M.

Suppose next that $\overline{G}$ is an alternating group. Let $N = \langle {P_0}^M \rangle \trianglelefteq M$. Then $\ker \theta|_N = N$ by Lemma 3.3, hence $N = P_0$ since θ is faithful on all other primes by Lemma 3.13. Thus $P_0 \trianglelefteq M$.

Finally, if $\overline{G}$ is a sporadic simple group, then the possibilities for $\overline{G}$ and p are given in Lemma 3.17. By direct inspection of the maximal subgroups of these groups (see [CCN+85]), the normalizer of P is the unique maximal subgroup containing $\Omega_1(P)$. □

We have established all of the conclusions of Proposition 3.1; the proof is complete.

Chapter 4

Sufficient Conditions for the Existence of Unfaithful Minimal Heilbronn Characters

In this chapter we complete the proof of Theorem 1 and establish as well Theorem 2 by proving a strong converse to Proposition 3.1:

Proposition 4.1. *Suppose G is a finite group with a Sylow p-subgroup P such that all of the following hold:*

(i) p is odd,

(ii) G is quasisimple with a cyclic center of order prime to p,

(iii) P is cyclic,

(iv) $N_G(P)$ is a maximal subgroup of G, and

(v) Either $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$, or $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$. (In the latter case $\Omega_1(P)$ is also contained in a Borel subgroup $N_G(Q)$ for some Sylow q-subgroup Q of G.)

Then G has a minimal Heilbronn character θ such that $\theta|_P$ is an unfaithful character of P. Moreover if P_1 is any nontrivial subgroup of P, then G has a minimal Heilbronn character θ with $\ker \theta|_P = P_1$.

4.1 Lemmas for Chapter 4

Lemma 4.2. *Suppose $N = P \rtimes H$ where P is an abelian p-group for some prime p. Then for any $t \in \mathbb{Z}$, the map $\varphi : N \to N$ defined by*

$$\varphi(n) = x^t h \qquad \text{where } n = xh,\ x \in P,\ h \in H$$

is a homomorphism and

$$\ker \varphi = \{x \in P \mid x^t = 1\}.$$

In particular φ is an automorphism of N if and only if $p \nmid t$.

Proof. Let $n_1, n_2 \in N$ with $n_1 = x_1 h_1$, $n_2 = x_2 h_2$, $x_i \in P$ and $h_i \in H$ for each $i = 1, 2$. Then

$$n_1 n_2 = x_1 h_1 x_2 h_2 = x_1 (h_1 x_2 h_1^{-1}) h_1 h_2, \tag{4.1}$$

and since $h_1 x_2 h_1^{-1} \in P$,

$$\begin{aligned} \varphi(n_1 n_2) &= \left(x_1 (h_1 x_2 h_1^{-1})\right)^t h_1 h_2 \\ &= x_1^t (h_1 x_2 h_1^{-1})^t h_1 h_2 \\ &= x_1^t (h_1 x_2^t h_1^{-1}) h_1 h_2 \\ &= x_1^t h_1 x_2^t h_2 \\ &= \varphi(n_1)\varphi(n_2). \end{aligned}$$

Hence φ is a homomorphism. Since $P \cap H = 1$, $\ker \varphi = \{x \in P \mid x^t = 1\} \leq P$. The final statement of the lemma follows from the observation that $x^t = 1$ for some nontrivial element $x \in P$ only if p divides t. □

Lemma 4.3. *Let P be an abelian p-group and h a p'-element normalizing P. Then every element of $[P, \langle h \rangle]$ is a simple commutator $[x, h]$ for some $x \in P$. If moreover h acts fixed point freely on P, then $P = [P, h]$.*

Proof. Let $\mathcal{A} = \{[x, h] \mid x \in P\}$ be the set of simple commutators in $[P, h]$. Define the map $\varphi : P \to \mathcal{A}$ by $\varphi(x) = [x, h]$ for any $x \in P$ (in fact φ is a homomorphism, but we will not need this).

By Fitting's Lemma, $P = C_P(h) \times [P, \langle h \rangle]$. We argue that φ maps $[P, \langle h \rangle]$ injectively into $\mathcal{A}$.

Let $a, b \in [P, \langle h \rangle]$, and suppose $\varphi(a) = \varphi(b)$. Then

$$[a, h] = [b, h] \quad \Longrightarrow \quad ba^{-1}h^{-1} = h^{-1}ba^{-1},$$

so $[ba^{-1}, h^{-1}] = 1$. Thus $ba^{-1} \in C_P(h) \cap [P, \langle h \rangle]$, so $a = b$. Then since φ is an injective map from $[P, \langle h \rangle]$ into the subset $\mathcal{A}$ of $[P, \langle h \rangle]$, in fact $[P, \langle h \rangle] = \mathcal{A}$ by order considerations. This proves the first statement.

It follows from the first statement that $[P, \langle h \rangle] = [P, h]$. If h acts fixed point freely on P, then $C_P(h) = 1$, which completes the proof. □

Corollary 4.4. *If P is a cyclic p-group and h is a p'-element normalizing but not centralizing P, then every element of P is a simple commutator of the form $[x, h]$ for some $x \in P$.*

Proof. Since P is cyclic and $P \neq C_P(h)$, $P = [P, \langle h \rangle]$ by Fitting's Lemma. The result now follows from Lemma 4.3. □

The following lemma collects some results from basic character theory, phrased to closely fit the needs of the proof of Proposition 4.1.

Lemma 4.5. *Suppose G is quasisimple, $g \in G$, $1 \neq \langle z \rangle = Z(G)$, and ψ is a virtual character of G. Suppose further that ψ restricts to a character of $\langle g \rangle Z(G)$, and that $\psi|_{Z(G)} = \chi|_{Z(G)}$ for some irreducible character χ of G. Then $\psi(zg) = \zeta \cdot \psi(g)$ for ζ an $|Z(G)|^{th}$ root of unity, primitive if and only if χ is faithful.*

Proof. Let Ψ be a matrix representation affording χ. Then $\Psi(z)$ is a scalar matrix by Schur's Lemma: $\Psi(z) = \zeta I$ where $\zeta \in \mathbb{C}$ and I is the identity matrix. If $n = |Z(G)|$, then $n > 1$ and $I = \Psi(z^n) = \Psi(z)^n = \zeta^n I$ implies $\zeta^n = 1$, i.e. ζ is an n^{th} root of unity. Since G is quasisimple, $\ker \chi \leq Z(G)$, and it follows that ζ is primitive if and only if χ is faithful. Finally, let $\widetilde{\Psi}$ be a representation affording ψ on $\langle g \rangle Z(G)$ (so in particular $\widetilde{\Psi}|_{Z(G)} = \Psi|_{Z(G)}$). Then $\widetilde{\Psi}(zg) = \widetilde{\Psi}(z)\widetilde{\Psi}(g) = \zeta\widetilde{\Psi}(g)$ implies $\psi(zg) = \zeta\psi(g)$ by considering the trace of the matrix $\zeta\widetilde{\Psi}(g)$. □

4.2 Proof of Proposition 4.1

Let $N = N_G(P)$, and write $N = P \rtimes H$ (by the Schur-Zassenhaus Theorem). For $x \in P$, $h \in H$, define $\varphi : N \to N$ by $\varphi(xh) = x^{|P_1|}h$ (as in Lemma 4.2).

Let π be any faithful irreducible character of G (such a character exists by Corollary 2.26). Define the map $\theta : G \to \mathbb{C}$ by

$$\theta(g) = \begin{cases} \pi(\varphi(n)) & \text{if } g \text{ is conjugate to } n \in N; \\ \pi(g) & \text{otherwise} \end{cases} \tag{4.2}$$

unless $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$, in which case define

$$\theta(g) = \begin{cases} \pi(\varphi(n)) & \text{if } g \text{ is conjugate to } n \in N; \\ \pi(1) & \text{if } |g| = q; \\ \pi(g) & \text{otherwise.} \end{cases} \tag{4.3}$$

We argue that, up to the addition of a nonnegative number of copies of the principal character 1_G of G (which we will consider shortly), θ is a minimal Heilbronn character of G with $\ker \theta|_P = P_1$. In the special case of (4.3), θ is unfaithful as well on an equi-characteristic Sylow subgroup of G.

Let $\Pi : G \to GL(V)$ be a representation of G affording the character π. Then by Lemma 4.2, $\Theta = \Pi|_N \circ \varphi$ is a representation of N into $GL(V)$ with $\ker \Theta = P_1$. The map θ defined in (4.2) or (4.3) extends the character of N afforded by Θ to a class function on G.

Let M be a maximal subgroup of G. We argue that $\theta|_M$ is a character of M. If M is conjugate to N, then $\theta|_M$ is precisely the character afforded by the representation Θ. Suppose then that M is not conjugate to N, and suppose further that N is the unique maximal subgroup of G containing $\Omega_1(P)$. In this case we argue that $\theta|_M = \pi|_M$. Let $m \in M$. If m is not conjugate to any element of N, then $\theta(m) = \pi(m)$. Otherwise, replacing M by a conjugate (which is permissible since θ is a class function on G), we may assume $m \in N$. Write $m = xh$ for $x \in P$, $h \in H$.

Suppose that x and h commute. Since φ is the identity on H,

$$\theta(h) = \pi(h) \textit{ for any } h \in H. \tag{4.4}$$

Hence if $x = 1$, then $m \in H$ and $\theta(m) = \pi(m)$. Otherwise p divides the order of m, so $\Omega_1(P) \leq M$, a contradiction.

Thus if $x \neq 1$, $[x, h] \neq 1$. Then $[x, h^{-1}] \neq 1$ as well, so $x = [y, h^{-1}]$ for some $y \in P$ by Corollary 4.4. Thus

$$x = [y, h^{-1}] = y^{-1}hyh^{-1},$$

which implies $xh = y^{-1}hy$, i.e. xh is conjugate to h in N. Since θ and π are class functions on N, it follows that $\theta(xh) = \theta(h)$ and $\pi(h) = \pi(xh)$. Thus $\theta(xh) = \pi(xh)$ (since $\theta(h) = \pi(h)$ by (4.4)), and in particular $\theta|_M = \pi|_M$ is a character of M.

It remains to consider the special case where $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$, and M is a Borel subgroup of G that contains $\Omega_1(P)$ and is not conjugate to N. Here M is a Frobenius group $Q \rtimes C$ for some equi-characteristic Sylow subgroup Q of G, and, replacing M by a conjugate if necessary, $C = M \cap N$ is cyclic of order $(q-1)/2$ (cf. Theorem 2.16). By hypothesis q is prime, so $|m| = q$ for all nontrivial $m \in Q$, and in particular θ is constant on Q. Since we have established that $\theta|_N$ is a character of N, $\theta|_C$ is a character of $C \leq N$. Thus $\theta|_M$ is a character of the Frobenius complement with the Frobenius kernel in its kernel, and is therefore a character of M.

We have shown that $\theta|_M$ is a character for every maximal subgroup M of G. It follows that θ restricts to a character of every elementary subgroup of G, and since θ is a G-class function, θ is a virtual character of G by Brauer's Characterization of Characters (Theorem 2.21). Moreover, θ is not a character of G since otherwise $1 \neq P_1 \leq \ker\theta \lhd G$ contradicts the hypothesis that G is quasisimple with p' center.

It remains to show that $\langle\theta, \mu\rangle \geq 0$ for every monomial character μ of G (i.e. θ is a Heilbronn character), and $\langle\theta, \chi\rangle \geq 0$ for every unfaithful irreducible character χ of G (completing the proof that θ is in fact a *minimal* Heilbronn character — cf. Lemma 1.2).

We consider first the inner product of θ with unfaithful irreducible characters χ of G, arguing that if G is not simple then $\langle\theta, \chi\rangle = 0$ for every such χ (we consider the case where G is simple and χ is the principal character separately). For any irreducible character χ of G (faithful or not),

$$\langle\theta, \chi\rangle = \frac{1}{|G|}\sum_{g\in G}\theta(g)\overline{\chi}(g) \tag{4.5}$$

$$= \frac{1}{|G|}\sum_{\overline{g}\in\overline{G}}\sum_{z\in Z(G)}\theta(\overline{g}z)\overline{\chi}(\overline{g}z) \tag{4.6}$$

where $\overline{G} = G/Z(G)$, and for $\overline{g}$ we choose a fixed representative in G of the coset $gZ(G)$. Let $Z(G) = \langle z\rangle$. Observing that $Z(G) \leq H$, $\theta|_{Z(G)} = \pi|_{Z(G)}$

by (4.4). It follows from Lemma 4.5 that

$$\theta(\overline{g}z) = \zeta \cdot \theta(\overline{g}) \quad \text{and} \quad \overline{\chi}(\overline{g}z) = \omega \cdot \overline{\chi}(\overline{g}) \tag{4.7}$$

for ζ a primitive $|Z(G)|^{\text{th}}$ root of unity and ω an $|Z(G)|^{\text{th}}$ root of unity that is primitive if and only if χ is faithful. Hence

$$\langle \theta, \chi \rangle = \frac{1}{|G|} \sum_{\overline{g} \in \overline{G}} \sum_{i=1}^{|Z(G)|} \zeta^i \theta(\overline{g})\, \omega^i \overline{\chi}(\overline{g}) \tag{4.8}$$

$$= \frac{1}{|G|} \sum_{\overline{g} \in \overline{G}} \theta(\overline{g}) \overline{\chi}(\overline{g}) \sum_{i=1}^{|Z(G)|} (\zeta\omega)^i. \tag{4.9}$$

If $\zeta\omega \neq 1$, then the sum over each fixed $\overline{g} \in \overline{G}$ is zero since $\zeta\omega$ is an $|Z(G)|^{\text{th}}$ root of unity. But if $|Z(G)| \neq 1$, then $\zeta\omega = 1$ only when $\omega = \zeta^{-1}$ is a primitive root, which occurs only when χ is faithful. Hence if G is not simple. then $\langle \theta, \chi \rangle = 0$ for every unfaithful character $\chi \in \mathrm{Irr}(G)$, as claimed.

We show finally that θ, perhaps with the addition of some copies of the principal character, is a Heilbronn character. Let μ be a monomial character of G. If μ is induced from a linear character of a proper subgroup H of G, then since $\theta|_H$ is a character, $\langle \theta, \mu \rangle \geq 0$ by Frobenius Reciprocity (cf. Lemma 1.1). Otherwise μ is a linear character of G, and since G is quasisimple and therefore perfect, μ is in fact the principal character 1_G of G. If G is not simple, then since 1_G is unfaithful, $\langle \theta, 1_G \rangle = 0$ by the argument in the preceding paragraph. Otherwise observe that we may add arbitrarily many copies of the principal character to θ without changing the properties established thus far: $\theta + n \cdot 1_G$ remains a virtual character but not a character of G that is a character restricted to every proper subgroup and has P_1 in its "kernel." Thus if $\langle \theta, 1_G \rangle < 0$, we redefine θ by adding enough copies of the principal character to overcome this obstruction. Hence θ is a Heilbronn character, completing the proof. □

Observe that we have proven:

Corollary 4.6. *Assume the hypotheses and notation of Proposition 4.1. Let π be any faithful irreducible character of G, and define the map $\theta : G \to \mathbb{C}$ as in (4.2) or (4.3) of the proof of that proposition. Then for some nonnegative integer k, $\widetilde{\theta} = \theta + k \cdot 1_G$ is a minimal Heilbronn character of G with $\ker \widetilde{\theta}|_P = P_1$. When G is not simple we may take $k = 0$, and in this case $\langle \theta, \chi \rangle = 0$ for every unfaithful character $\chi \in Irr(G)$.*

Chapter 5

Additional Results on Minimal Heilbronn Characters

5.1 Groups Possessing Unfaithful Minimal Heilbronn Characters

Theorem 1 completely classifies the groups possessing unfaithful minimal Heilbronn characters in terms of their *properties*; in this chapter we begin to determine precisely when these properties occur. A complete answer to this question would require an exact knowledge of the maximal subgroups of each candidate family of quasisimple groups. Much work has been done on this subject, including the O'Nan-Scott Theorem (which we will not need) classifying the maximal subgroups of the symmetric — and therefore the alternating — groups, and the work of Kleidman and Liebeck ([KL90]) classifying the "natural" subgroups of the classical groups of Lie type. Both of these classifications allow for the occurrence of almost simple maximal subgroups that arise irregularly, complicating the task of determining when $\Omega_1(P)$ is contained uniquely in $N_G(P)$. Thus, although we are able to give precise conditions describing when the alternating groups possess unfaithful minimal Heilbronn characters, for the linear, unitary, and symplectic groups we are only able to give necessary conditions. We will confine our investigations in this chapter to these groups, as well as the sporadic simple groups (for which we already have a complete answer), leaving the orthogonal and exceptional groups of Lie type for future research.

We will prove the following theorem, which strengthens Theorem 3 (in Section 1.3):

Theorem 5.1. *Let G be a finite quasisimple group.*

(1) If $G/Z(G)$ is an alternating group, then G possesses an unfaithful minimal Heilbronn character if and only if G has prime degree $p \neq 11$ or 23 and the cyclotomic polynomial $\Phi_d(q) \neq p$ for any prime d and prime power q. When this occurs the minimal Heilbronn character is unfaithful precisely on the prime p.

(2) If $G/Z(G)$ is a sporadic simple group, then G possesses an unfaithful minimal Heilbronn character if and only if $G/Z(G)$ is isomorphic to J_1, M_{23}, Ly, J_4, Fi'_{24}, or B. When this occurs, the primes on which the minimal Heilbronn character may be unfaithful are specified in Lemma 3.17.

(3) If $G/Z(G)$ is isomorphic to $L_n(q)$, $U_n(q)$, or $PSp_{2n}(q)$, then G possesses an unfaithful minimal Heilbronn character only if n is prime.

Proof. The result for the sporadic groups has already been proven — see Lemma 3.17.

If $G/Z(G)$ is an alternating group of degree n, then by Theorem 1 and Lemma 3.13, G possesses an unfaithful minimal Heilbronn character precisely when $n = p$ is prime and $N_G(P)$ is the unique maximal subgroup containing P. Suppose $M \neq N_G(P)$ is maximal in G and contains P. Since P is not normal in M, M is 2-transitive by a classical theorem of Burnside (Theorem 2.12). By Proposition 2.13, M has a unique minimal normal subgroup T, and T is either simple or elementary abelian.

Suppose $T \trianglelefteq M$ is elementary abelian. Then $H = TP$ is a transitive subgroup of M (since P itself is). It follows that the stabilizer of a point has index p in H, and as T is the unique such subgroup, T fixes every point. Hence T is the identity, a contradiction.

We have established that T is simple. All such simple groups and the degrees of the permutation groups in which they are doubly transitive are listed in Table 2.6. Using the fact that in each case the specified degree must equal p, we proceed to eliminate all possibilities except for the groups $L_d(q)$ and M_{23}.

The observation that q^3+1 factors, hence cannot equal p, eliminates the unitary and Ree groups. The symplectic groups occur as doubly transitive minimal normal subgroups only in permutation groups of even degree, so T is not symplectic. Finally, T is not a Suzuki group ${}^2B_2(q)$ since it is well known that 2^n+1 is an odd prime only when n is a power of 2, hence $q = 2^{2k+1}$ implies $q^2+1 \neq p$.

The remaining cases do indeed occur: $T \cong L_2(11) < M_{11} < A_{11}$, $T \cong M_{23} < A_{23}$, and

$$T \cong L_d(q) < A_p \quad \text{if and only if} \quad \frac{q^d-1}{q-1} = \Phi_d(q) = p. \tag{5.1}$$

Since $\Phi_d(q)$ factors when d is composite, $\Phi_d(q) = p$ only if d is prime.

Assume next that G is a classical group of Lie type possessing a minimal Heilbronn character that is unfaithful on a Sylow p-subgroup P of G, and if $G/Z(G) \cong L_n(q)$ then assume that $n > 2$ (since 2 is prime, the theorem holds when $n = 2$). Observe that p does not divide the order of any parabolic subgroup of G by Lemma 3.21.

We argue first that n must be prime when $G/Z(G) \cong L_n(q)$. Here the order of G is a power of q times the cyclotomic polynomial $\Phi_n(q)$ and smaller degree cyclotomic factors, possibly adjusted by some p' factor of the Schur multiplier. It follows that p divides $\Phi_n(q)$, since if p divides one of the smaller cyclotomic factors, then p divides the order of a parabolic subgroup. Thus p divides q^n-1 (since $\Phi_n(q)$ does), and we may assume without loss of generality that p is a Zsigmondy prime (cf. Theorem 2.22), i.e. p does not divide q^i-1 for any $i < n$. Writing $n = ms$ for s a prime, G has a subgroup $H \cong GL_m(q^s)$ by Theorem 2.17. The order of H is $q^{ms}-1$ times other factors (if $m > 1$), so p divides $|H|$, and in particular $\Omega_1(P) \leq H$. Since $|Z(H)| = q^s-1$ and p is Zsigmondy, P is contained in the center of H if and only if $s = ms$, i.e. $m = 1$ and $n = s$ is prime. Since $P \trianglelefteq H$ by Theorem 1 and a normal p-subgroup of H is necessarily contained in $Z(H)$, n must be prime.

If $G/Z(G) \cong SP_{2n}(q)$ with $n \geq 2$, then by similar reasoning we have p a Zsigmondy prime dividing $q^{2n}-1$. Writing $n = ms$ for s a prime, G has a subgroup $H \cong Sp_{2m}(q^s)$ by Theorem 2.17. As in the previous case, $P \trianglelefteq H$ forces $m = 1$ and $n = s$.

Now suppose $G/Z(G) \cong U_n(q)$ with $n \geq 3$. We argue first that n is odd.

For otherwise

$$|G| = q^{n(n-1)/2}(q^n - 1)\cdots(q^{n/2}+1)\cdots(q^3+1)(q^2-1),$$

possibly adjusted by some p' factor of the Schur multiplier. Since p does not divide the order of any parabolic subgroup, we must have $p|(q^n-1)$ but $p \nmid (q^{n/2}+1)$, which implies $p \mid (q^{n/2}-1)$. By Theorem 2.17, G has a subgroup $H \cong GL_{n/2}(q^2)$, and H contains $\Omega_1(P)$ by consideration of its order. Taking p to be Zsigmondy with respect to $q^{n/2}-1$ ensures that P is not contained in the center of H, hence P is not normal in H, a contradiction — $\Omega_1(P)$ is contained in a maximal subgroup other than $N_G(P)$.

Thus n is odd and

$$|G| = q^{n(n-1)/2}(q^n+1)(q^{n-1}-1)\cdots(q^3+1)(q^2-1),$$

again with some possible adjustment by a p' factor of the Schur multiplier. We may take p dividing q^n+1 but no other factor by taking a Zsigmondy prime of $q^{2n}-1 = (q^n-1)(q^n+1)$. Writing $n = ms$ for s a prime, G has a subgroup $H \cong GU_m(q^s)$ by Theorem 2.17, and, arguing as in the linear and symplectic cases, it follows that n must be prime. □

Remark. The orthogonal groups are not considered in Theorem 5.1 because they resist an analogous treatment. For example, consider the case where $G \cong \Omega_n(q)$, q is odd, and n is both odd and composite. Then there exists a subgroup $H \cong \Omega_{n/r}(q^r)$ of G for every prime divisor r of n [KL90, Table 3.5.D]. However, a Zsigmondy prime p chosen to avoid the parabolic subgroups of G does not divide the order of H, so we cannot proceed as we did for the linear, symplectic, and unitary groups.

We can use Theorem 5.1 to compute directly whether or not the alternating group A_p possesses an unfaithful minimal Heilbronn character for a given prime p. The results for $p < 1000$ are listed in Table 5.1. For those primes not listed (except the special cases $p = 11$ and $p = 23$), A_p contains a p-singular simple subgroup $L_d(q)$ for some prime d and prime power q. In particular:

$L_3(2) < A_7$,	$M_{11} < A_{11}$,	$L_3(3) < A_{13}$,	$L_2(16) < A_{17}$,
$M_{23} < A_{23}$,	$L_3(5) < A_{31}$,	$L_3(8) < A_{73}$,	$L_7(2) < A_{127}$
$L_2(256) < A_{257}$,	$L_3(17) < A_{307}$,	$L_3(27) < A_{757}$.	

Table 5.1: Primes $p < 1000$ such that
A_p has an Unfaithful Minimal Heilbronn Character

5	19	29	37	41	43	47	53	59	61
67	71	79	83	89	97	101	103	107	109
113	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	263	269	271	277	281	283
293	311	313	317	331	337	347	349	353	359
367	373	379	383	389	397	401	409	419	421
431	433	439	443	449	457	461	463	467	479
487	491	499	503	509	521	523	541	547	557
563	569	571	577	587	593	599	601	607	613
617	619	631	641	643	647	653	659	661	673
677	683	691	701	709	719	727	733	739	743
751	761	769	773	787	797	809	811	821	823
827	829	839	853	857	859	863	877	881	883
887	907	911	919	929	937	941	947	953	967
971	977	983	991	997					

5.2 Groups Possessing Minimal Heilbronn Characters

We present here a proof of the general classification in Theorem 4, which we restate for the convenience of the reader:

Theorem 4. *A finite group G possesses a minimal Heilbronn character if and only if there exists a faithful, irreducible, nonlinear, primitive character χ of G such that whenever χ is a constituent of ψ^* for some irreducible character ψ of a maximal subgroup of G, some irreducible character $\chi' \neq \chi$ of G is also a constituent of ψ^*.*

Proof. Let θ be a minimal Heilbronn character of G, so $\langle \theta, \chi \rangle < 0$ for some faithful, irreducible, nonlinear, primitive character χ of G by Lemma 1.6. Suppose, to obtain a contradiction, that there is some irreducible character ψ of some maximal subgroup M such that χ is the unique irreducible constituent of ψ^*. By Frobenius Reciprocity $\langle \chi'|_M, \psi \rangle = 0$ for all irreducible characters $\chi' \neq \chi$ of G. It follows that $\langle \theta|_M, \psi \rangle = \langle \theta, \chi \rangle \cdot \langle \chi|_M, \psi \rangle < 0$, so $\theta|_M$ is not a character of M, a contradiction.

For the converse, suppose χ is a faithful, irreducible, nonlinear, primitive character of G for which the stated condition holds. For any character ψ of any maximal subgroup M such that χ is a constituent of ψ^*, let χ' denote an irreducible character distinct from χ that is also a constituent of ψ^*, and set a equal to the smallest integer that is at least as large as the quotient of $\langle \chi|_M, \psi \rangle$ by $\langle \chi'|_M, \psi \rangle$. Let $\mathcal{M}$ be a set of representatives of the conjugacy classes of maximal subgroups of G. We argue that

$$\theta = \left(\sum_{M \in \mathcal{M}} \sum_{\psi \in \mathrm{Irr}(M)} a\chi' \right) - \chi \tag{5.2}$$

is a minimal Heilbronn character of G.

Clearly θ is a virtual character of G, and since each character χ' is distinct from χ, $\langle \theta, \chi \rangle = -1$ and $\langle \theta, \chi' \rangle \geq 0$ for all other irreducible constituents. In particular, θ is a Heilbronn character (since χ is not monomial) but not a character of G. Thus by Lemma 1.2 it remains to show that θ restricts to a character of every proper subgroup of G, and in fact it suffices to consider only the maximal subgroups of G.

Let M be a maximal subgroup of G and $\psi \in \mathrm{Irr}(M)$. Let $\theta = \varphi - \chi$ where φ is the double sum in (5.2). By construction φ includes a constituent $a\chi'$ where χ' is a constituent of ψ^* distinct from χ, hence

$$\begin{aligned}
\langle \theta|_M, \psi\rangle &= \langle \varphi|_M, \psi\rangle - \langle \chi|_M, \psi\rangle && (5.3)\\
&\geq a\langle \chi'|_M, \psi\rangle - \langle \chi|_M, \psi\rangle && (5.4)\\
&\geq \left(\frac{\langle \chi|_M, \psi\rangle}{\langle \chi'|_M, \psi\rangle}\right)\langle \chi'|_M, \psi\rangle - \langle \chi|_M, \psi\rangle && (5.5)\\
&= 0, && (5.6)
\end{aligned}$$

where (5.5) follows from (5.4) by our choice of the constant a. Hence $\theta|_M$ is a character, which completes the proof. □

Remark. The minimal Heilbronn character θ constructed in the proof of Theorem 4 is in general of excessively large degree. An iterative technique, replacing θ for each successive character ψ of each maximal subgroup M and then adding only as many copies of the character χ' as needed to force $\langle \theta|_M, \psi\rangle \geq 0$, would in general produce a minimal Heilbronn character of smaller degree. For such a technique the degree would depend on both the order through which the maximal subgroups and their characters were iterated, and the characters χ' chosen to "neutralize" the restriction of the negative constituent χ. It seems unlikely that these choices could be codified into an optimal algorithm that always produced a minimal Heilbronn character of smallest possible degree.

Chapter 6

Characterization of the Exceptional Case $L_2(q)$

The groups $L_2(q)$ with q prime are unique among the groups of Lie type in that their equi-characteristic Sylow subgroups are cyclic, and it is precisely this property that distinguishes them in the context of unfaithful minimal Heilbronn characters. In this chapter we apply Corollary 4.6 to the construction of unfaithful minimal Heilbronn characters from nonprincipal irreducible characters of $L_2(q)$, both in the general case where $\Omega_1(P)$ is contained only in its own normalizer and in the special case where $\Omega_1(P)$ is contained as well in a Borel subgroup.

6.1 Lemmas for Chapter 6

We require the complex character tables for the groups $L_2(q)$. If q is even, then $L_2(q) = SL_2(q)$ (as explained in Section 2.1.4), so Table 2.7 gives the character table in this case. We consider the situation when q is odd in the following proposition.

Proposition 6.1. *Suppose $G \cong L_2(q)$ for $q \geq 5$ a power of an odd prime. Then Table 6.1 or 6.2 is the complex character table for G, according to whether $q \equiv 1 \bmod 4$ or $q \equiv 3 \bmod 4$, respectively. In each table the column headings specify representatives of the distinct conjugacy classes followed by the size of the class, and all other notation is as specified following the table.*

Table 6.1: Characters of $L_2(q)$, $q \equiv 1 \bmod 4$

class:	1	t	a^l	b^m	c	d
size:	1	$\frac{1}{2}q(q+1)$	$q(q+1)$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$
1_G	1	1	1	1	1	1
ξ_1	$\frac{1}{2}(q+1)$	$(-1)^{\frac{q-1}{4}}$	$(-1)^l$	0	$\frac{1}{2}\left(1+\sqrt{q}\right)$	$\frac{1}{2}\left(1-\sqrt{q}\right)$
ξ_2	$\frac{1}{2}(q+1)$	$(-1)^{\frac{q-1}{4}}$	$(-1)^l$	0	$\frac{1}{2}\left(1-\sqrt{q}\right)$	$\frac{1}{2}\left(1+\sqrt{q}\right)$
ψ_i	$q-1$	0	0	$-(\zeta^{im}+\zeta^{-im})$	-1	-1
σ	q	1	1	-1	0	0
χ_j	$q+1$	$2(-1)^j$	$\rho^{jl}+\rho^{-jl}$	0	1	1

$$|t| = 2, \quad |a| = (q-1)/2, \quad |b| = (q+1)/2, \quad |c| = |d| = q,$$
$$1 \le i, m \le (q-1)/4, \quad 1 \le j, l \le (q-5)/4,$$
$$\rho^{\frac{1}{2}(q-1)} = \zeta^{\frac{1}{2}(q+1)} = 1, \text{ primitive roots of unity in } \mathbb{C}.$$

Table 6.2: Characters of $L_2(q)$, $q \equiv 3 \bmod 4$

class:	1	t	a^l	b^m	c	d
size:	1	$\frac{1}{2}q(q-1)$	$q(q+1)$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$
1_G	1	1	1	1	1	1
η_1	$\frac{1}{2}(q-1)$	$(-1)^{\frac{q+5}{4}}$	0	$(-1)^{m+1}$	$-\frac{1}{2}\left(1-\sqrt{-q}\right)$	$-\frac{1}{2}\left(1+\sqrt{-q}\right)$
η_2	$\frac{1}{2}(q-1)$	$(-1)^{\frac{q+5}{4}}$	0	$(-1)^{m+1}$	$-\frac{1}{2}\left(1+\sqrt{-q}\right)$	$-\frac{1}{2}\left(1-\sqrt{-q}\right)$
ψ_i	$q-1$	$2(-1)^{i+1}$	0	$-(\zeta^{im}+\zeta^{-im})$	-1	-1
σ	q	-1	1	-1	0	0
χ_j	$q+1$	0	$\rho^{jl}+\rho^{-jl}$	0	1	1

$$|t| = 2, \quad |a| = (q-1)/2, \quad |b| = (q+1)/2, \quad |c| = |d| = q,$$
$$1 \le i, j, l, m \le (q-3)/4,$$
$$\rho^{\frac{1}{2}(q-1)} = \zeta^{\frac{1}{2}(q+1)} = 1, \text{ primitive roots of unity in } \mathbb{C}.$$

Proof. The complex character table for $SL_2(q)$ is given in Table 2.8. By Lemma 2.24, the corresponding characters for $L_2(q)$ are those with the central conjugacy class, z, in the kernel. These can be read directly from Table 2.8, and depend on the congruence of q modulo 4 (which determines the value of ϵ). We see that the characters ψ_i and χ_j occur precisely when i and j are even, σ occurs always, ξ_1 and ξ_2 occur if and only if $q \equiv 1 \bmod 4$, and η_1 and η_2 occur if and only if $q \equiv 3 \bmod 4$. Taking even subscripts of ψ_i for $1 \leq i \leq (q-1)/2$ is equivalent to taking all subscripts of ψ_i for $1 \leq i \leq (q-1)/4$ after replacing ζ by its square, a primitive $\frac{1}{2}(q+1)$'th root of unity, and similarly we take all subscripts of χ_j for $1 \leq j \leq (q-3)/4$ and replace ρ by a primitive $\frac{1}{2}(q-1)$'th root of unity. Adjusting the upper bounds on i and j to ensure integrality, we get the set of characters in Tables 6.1 and 6.2.

It is clear that the $SL_2(q)$ conjugacy classes z, zc, and zd do not occur in $L_2(q)$, being absorbed into the classes 1, c, and d, respectively. The remaining $SL_2(q)$ classes a^l, b^m, c, and d maintain their sizes in $L_2(q)$ since the orders of both the group and the centralizers of representatives decrease by the same factor of $2 = |Z(G)|$.

By Theorem 2.16, G has dihedral subgroups of order $q-1$ and $q+1$. The cyclic components of these subgroups are generated by the elements a and b, respectively. Since precisely one of $(q \pm 1)/2$ is even, precisely one of these cyclic subgroups contributes a conjugacy class of involutions. In particular, if $q \equiv 1 \bmod 4$, then $t = a^{(q-1)/4}$ is an involution, and if $q \equiv 3 \bmod 4$, then $t = b^{(q+1)/4}$ is an involution. The values of the irreducible characters of G on t are thus determined by their values on the specified power of a or b. Observe in particular that $\rho^{(q-1)/4} = \zeta^{(q+1)/4} = -1$, justifying the values of $\chi_j(t)$ (when $q \equiv 1 \bmod 4$) and $\psi_i(t)$ (when $q \equiv 3 \bmod 4$). Since the centralizer of t is precisely the cyclic subgroup $\langle a \rangle$ or $\langle b \rangle$ in which it resides, the size of the conjugacy class of involutions is $q(q+1)/2$ or $q(q-1)/2$ according to whether $q \equiv 1 \bmod 4$ or $q \equiv 3 \bmod 4$, respectively.

Observing that a^l and $a^{-l} = a^{l(q-3)/2}$ are conjugate in $L_2(q)$ (by the usual action of the dihedral group on its cyclic subgroup), the conjugacy classes of a^l run from $l = 1$ to $l = (q-3)/4$ instead of $(q-3)/2$, as they do in $SL_2(q)$. We must remove the last class $a^{(q-3)/4}$ in the case where $q \equiv 1 \bmod 4$, as this is the class t of involutions. Similarly, the classes b^m run from $m = 1$ to $m = (q-1)/4$, with the involution class removed when $q \equiv 3 \bmod 4$. All aspects of Tables 6.1 and 6.2 have now been justified, completing the proof. □

Lemma 6.2. *Let ζ be a primitive n^{th} root of unity, and fix a positive integer s. Let N be the greatest integer strictly less than $\frac{n}{2}$. Then*

$$\sum_{k=1}^{N}\left(\zeta^{sk}+\zeta^{-sk}\right)=\begin{cases}-1 & \text{if } s\not\equiv 0 \bmod n \text{ and } n \text{ is odd;}\\ -1+(-1)^{s+1} & \text{if } s\not\equiv 0 \bmod n \text{ and } n \text{ is even;}\\ n-1 & \text{if } s\equiv 0 \bmod n \text{ and } n \text{ is odd;}\\ n-2 & \text{if } s\equiv 0 \bmod n \text{ and } n \text{ is even.}\end{cases}$$

Proof. Since $\zeta^{sn}=1$, $\zeta^{-sk}=\zeta^{s(n-k)}$. Thus we may write

$$\sum_{k=1}^{N}\zeta^{-sk}=\sum_{k=N}^{1}\zeta^{s(n-k)}=\sum_{k=n-N}^{n-1}\zeta^{sk}.$$

Observing that $n-N$ is $N+1$ when n is odd and $N+2$ when n is even,

$$\sum_{k=1}^{N}\left(\zeta^{sk}+\zeta^{-sk}\right)=\sum_{k=1}^{n-1}\zeta^{sk}-\begin{cases}0 & \text{if } n \text{ is odd;}\\ \zeta^{s(N+1)} & \text{if } n \text{ is even.}\end{cases}$$

If $s\not\equiv 0 \bmod n$, then ζ^s is a nontrivial n^{th} root of unity, so $\sum_{k=1}^{n-1}\zeta^{sk}=-1$. Since $N+1=\frac{n}{2}$ when n is even, $\zeta^{N+1}=-1$, proving the first two cases. If instead $s\equiv 0 \bmod n$, then $\sum_{k=1}^{n-1}\zeta^{sk}=n-1$ and $\zeta^{s(N+1)}=1$, completing the proof. □

By inspection of Tables 6.1, 6.2, and 2.7, it is easily seen that the number of conjugacy classes $[a^l]$ and $[b^m]$ is the greatest integer strictly less than $|a|/2$ and $|b|/2$, respectively. We may therefore apply Lemma 6.2 to sums of the characters χ_j and ψ_i over the classes $[a^l]$ and $[b^m]$, respectively, as the character values on these classes involve $|a|^{\text{th}}$ and $|b|^{\text{th}}$ roots of unity.

Corollary 6.3. *Let $G\cong L_2(q)$, and assume the notation of the appropriate complex character table, Table 6.1, 6.2, or 2.7. Then if π is any nonprincipal irreducible character of G,*

$$\sum_{l=1}^{A}\pi(a^l)\le 2 \quad \textit{and} \quad \sum_{m=1}^{B}\pi(b^m)\le 2$$

where A and B are the number of conjugacy classes $[a^l]$ and $[b^m]$, respectively.

Proof. The result follows from Lemma 6.2 and inspection of the character tables. $\square$

Lemma 6.4. *Assume the notation of Corollary 4.6 and Tables 6.1, 6.2, and 2.7. Let $\mathcal{N}$ denote the set of elements of G on which θ and π potentially differ. Then*

$\mathcal{N}$ is the set of classes $[b^m]$ in case (i),

$\mathcal{N}$ is the set of classes $[c]$ and $[d]$ in case (ii), and

$\mathcal{N}$ is the set of classes $[a^l]$, $[c]$, and $[d]$ in case (iii).

Proof. Aside from the class $[t]$, these are the classes involved in N, along with the classes corresponding to the characteristic prime in case (iii). The class $[t]$ is not contained in $\mathcal{N}$ since any involution in N is G-conjugate to an involution in H, and φ is the identity on H. $\square$

Lemma 6.5. *In the notation of Corollary 4.6 and Tables 6.1, 6.2, and 2.7, there exist $\alpha, \beta \in \mathbb{N}$ such that $\varphi(b^m) = b^{\beta m}$ (in case (i)) and $\varphi(a^l) = a^{\alpha l}$ (in case (iii)).*

Proof. The dihedral group N has a characteristic cyclic subgroup C generated by b in case (i) and by a in case (iii). In either case φ maps C into C, hence $\varphi(b) = b^\beta$ for some constant β, and similarly $\varphi(a) = a^\alpha$. The result follows. $\square$

In fact the constants α and β are easily computed. In case (i), observe that $\langle b \rangle = P \times K$ for some cyclic subgroup K of H, so we may write $b = xh$ for $x \in P$ and $h \in K$. Then

$$\varphi(b) = x^{|P_1|}h = b^\beta = x^\beta h^\beta$$

as in Lemma 6.5. Observing that $|P|$ and $|K|$ are coprime, we may choose

$$\beta \equiv |P_1| \bmod |P| \quad \text{and} \quad \beta \equiv 1 \bmod |K|$$

by the Chinese Remainder Theorem. The computation of α in case (iii) is identical.

Lemma 6.6. *In the notation of Corollary 4.6, P_1 contains representatives of precisely $(|P_1| - 1)/2$ distinct nontrivial G-conjugacy classes.*

Proof. By a basic result due to Burnside (which follows from Sylow's Theorem), $x, y \in P_1$ are G-conjugate precisely when they are conjugate in N. Since N is dihedral and P_1 is contained in the characteristic cyclic subgroup of N, this occurs if and only if $x = y$ or an element of order 2 in N conjugates x into y, if and only if $x = y$ or $x = y^{-1}$. Hence there are exactly two elements of P_1 in each nontrivial conjugacy class, yielding $(|P_1| - 1)/2$ such classes. □

6.2 Unfaithful Minimal Heilbronn Characters of $L_2(q)$

We proceed to specify precisely when $L_2(q)$ possesses unfaithful minimal Heilbronn characters, and to describe their construction. In particular we show that in this case the nonnegative integer k in Corollary 4.6 may be taken to be zero.

Proposition 6.7. *Let $G \cong L_2(q)$ where $q = r^a \geq 5$ and r is a prime. Let $p > 5$ be a prime dividing the order of G, and suppose that one of the following holds:*

(i) p divides $q + 1$ but p does not divide $r^b \pm 1$ for any proper divisor b of a;

(ii) $p = q = r$ is an odd prime; or

(iii) p divides $q - 1$ and $q = r$ is an odd prime.

Let P be a Sylow p-subgroup of G, P_1 be any nontrivial subgroup of P, and $N = N_G(P) = P \rtimes H$. For $x \in P$, $h \in H$, define $\varphi : N \to N$ by $\varphi(xh) = x^{|P_1|}h$. Then if π is any nonprincipal irreducible character of G, the map $\theta : G \to \mathbb{C}$ given by

$$\theta(g) = \begin{cases} \pi(\varphi(n)) & \text{if } g \text{ is conjugate to } n \in N; \\ \pi(1) & \text{if case (iii) applies and } |g| = q = r; \\ \pi(g) & \text{otherwise} \end{cases} \tag{6.1}$$

is a minimal Heilbronn character of G with $\ker \theta|_P = P_1$. When case (iii) applies, θ is unfaithful as well on a Sylow r-subgroup of G.

Proof. The proposition follows from the proof of Proposition 4.1 once we establish:

(1) the equivalence of the stated hypotheses with those phrased in terms of maximal subgroups containing $\Omega_1(P)$, and

(2) that no copies of the principal character 1_G need to be added to θ in order to obtain $\langle \theta, 1_G \rangle \geq 0$.

We refer throughout to the classification of the maximal subgroups of $L_2(q)$ given in Theorem 2.16.

Since $|G| = q(q+1)(q-1)(2,q)/2$, p divides q, $q+1$, or $q-1$. If $p = 3$ or 5, then P is contained in a subgroup isomorphic to A_4 or A_5, respectively. The conditions in (i) ensure that p does not divide the order of any proper subgroup $PGL_2(r^b)$ or $PSL_2(r^b)$, and these subgroups do not occur in cases (ii) and (iii). Since $N_G(P)$ is maximal, (1) follows.

For (2), let $\mathcal{N}$ be the set of elements of G on which θ and π potentially differ (cf. Lemma 6.4). Then

$$\begin{aligned}\langle \theta, 1_G \rangle &= \frac{1}{|G|}\left(\sum_{g \in \mathcal{N}} \theta(g) + \sum_{g \notin \mathcal{N}} \pi(g)\right) \\ &= \langle \pi, 1_G \rangle + \frac{1}{|G|}\left(\sum_{g \in \mathcal{N}} \theta(g) - \sum_{g \in \mathcal{N}} \pi(g)\right) \\ &= \frac{1}{|G|}\sum_{g \in \mathcal{N}} \left(\theta(g) - \pi(g)\right).\end{aligned}$$

Since $\langle \theta, 1_G \rangle$ is integral, $\langle \theta, 1_G \rangle \geq 0$ if and only if $\langle \theta, 1_G \rangle > -1$, hence it suffices to prove:

$$\sum_{g \in \mathcal{N}} \left(\pi(g) - \theta(g)\right) < |G|. \qquad (6.2)$$

Referring to the complex character tables for $L_2(q)$ (Tables 6.1, 6.2, and 2.7), we proceed by cases.

Case *(i)* (p divides $q+1$):

Here $\mathcal{N}$ is the set of conjugacy classes $[b^m]$. Let B denote the number of such classes. Since each has size $q(q-1)$,

$$\begin{aligned}\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) &= q(q-1)\left(\sum_{m=1}^{B}\pi(b^m)-\sum_{m=1}^{B}\theta(b^m)\right)\\ &\le q(q-1)\left(2-\sum_{m=1}^{B}\theta(b^m)\right),\end{aligned}$$

by Corollary 6.3. Since $|\theta(b^m)|\le 2$ for all values of m such that $b^m\notin P_1$, and $b^m\in P_1$ for at least one value of m,

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g))\le q(q-1)\left(2-\pi(1)+2B\right).$$

Observing that $2-\pi(1)<0$

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g))<q(q-1)(2M)$$

and that $B\le |G|/2(q-1)(q+1)$,

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g))<\frac{q}{q+1}|G|.$$

This establishes (6.2) in case (i).

Case (*ii*) ($p=q=r$ is an odd prime):

Here there $\mathcal{N}$ consists only of the classes $[c]$ and $[d]$, both of size q^2-1. These classes are in the kernel of θ (since $P_1=P$), hence

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g))=(q^2-1)(\pi(c)-\pi(1))+(q^2-1)(\pi(d)-\pi(1)).$$

Since $\pi(1)>\pi(c)$ and $\pi(1)>\pi(d)$ for all nonprincipal $\pi\in\mathrm{Irr}(G)$, the sum above is negative and (6.2) follows.

Case (*iii*) (p divides $q-1$ and $q=r$ is an odd prime):

Here $\mathcal{N}$ is comprised of the conjugacy classes $[a^l]$ as well as the classes $[c]$ and $[d]$. Reasoning as in case (i), the sum over the classes $[a^l]$ is less than $|G|$, and we have seen that the sum over the classes $[c]$ and $[d]$ is negative. Hence (6.2) holds in this case as well, completing the proof. □

6.3 Norms of Unfaithful Minimal Heilbronn Characters of $L_2(q)$

For any virtual character ψ, the *norm* of ψ is defined (in terms of its square) as $||\psi||^2 = \langle \psi, \psi \rangle$. If we write ψ as a $\mathbb{Z}$-linear combination of distinct irreducible constituents, then $||\psi||^2$ is equal to the sum of the squares of the coefficients. Hence the norm provides a measure of how far a virtual character is from being irreducible, with $||\psi||^2 = 1$ if and only if ψ is plus or minus an irreducible character.

We proceed to calculate the norms of the minimal Heilbronn characters constructed in Proposition 6.7.

Proposition 6.8. *In the notation of Proposition 6.7, Tables 6.1, 6.2, and 2.7, and Lemma 6.5 (which defines α and β), the norm of θ is as specified in Table 6.3, with*

$$\delta = \begin{cases} 1 & \text{if } \pi = \psi_i \text{ and } i\beta \equiv 0 \bmod (2,q)(q+1)/4; \\ 1 & \text{if } \pi = \chi_j \text{ and } j\alpha \equiv 0 \bmod (q-1)/4; \\ 0 & \text{otherwise.} \end{cases}$$

π	$\|\|\theta\|\|^2$ case (i)	$\|\|\theta\|\|^2$ case (ii)	$\|\|\theta\|\|^2$ case (iii)
η_i	$\frac{1}{4}(q-3)(\|P_1\|-1)+1$	$\frac{1}{2}(q+1)$	$\frac{1}{4}(q-1)(\|P_1\|+1)+1$
ξ_i	$\frac{1}{4}(q+1)(\|P_1\|-1)+1$	$\frac{1}{2}(q+3)$	$\frac{1}{4}(q+3)(\|P_1\|+1)$
ψ_i	$\frac{1}{(2,q)}(q-3)(\|P_1\|-1)+1+\delta$	$2q-3$	$(q-1)(\|P_1\|-1)+2q-3$
σ	$\frac{1}{(2,q)}(q-1)(\|P_1\|-1)+1$	$2q+1$	$(q+1)(\|P_1\|-1)+2q+1$
χ_j	$\frac{1}{(2,q)}(q+1)(\|P_1\|-1)+1$	$2q+5$	$(q+3)(\|P_1\|-1)+2q+5+\delta$

Table 6.3: Norms of Unfaithful Minimal Heilbronn Characters of $L_2(q)$

Proof. Let $\mathcal{N}$ denote the set of elements of G on which θ and π may potentially differ (cf. Lemma 6.4). Then

$$
\begin{aligned}
||\theta||^2 &= \frac{1}{|G|}\left(\sum_{g\in\mathcal{N}}|\theta(g)|^2+\sum_{g\notin\mathcal{N}}|\pi(g)|^2\right)\\
&= ||\pi||^2+\frac{1}{|G|}\sum_{g\in\mathcal{N}}\left(|\theta(g)|^2-|\pi(g)|^2\right)\\
&= 1+\frac{1}{|G|}\sum_{g\in\mathcal{N}}\left(|\theta(g)|^2-|\pi(g)|^2\right). \qquad (6.3)
\end{aligned}
$$

We proceed by cases.

Case *(i)* (p divides $q+1$):

Here $\mathcal{N}$ is the set of classes $[b^m]$, each of size $q(q-1)$. By Lemma 6.6, $\theta(b^m)=\pi(1)$ for $(|P_1|-1)/2$ such classes. Writing b as the product of commuting p- and p'-elements x and y (respectively), $\varphi(b^m)$ is an element of order two precisely when $x\in P_1$ and $|y|=2$. If $|b|$ is odd, or equivalently if $q\equiv 0,1 \bmod 4$, then there is no such element. Otherwise $q\equiv 3 \bmod 4$ and there are $(|P_1|-1)/2$ classes with $\theta(b^m)=\pi(t)$.

Suppose first that π is not one of the characters ψ_i, so π is constant in absolute value across the classes $[b^m]$. In particular, if $\varphi(b^m)\neq 1$ and $\varphi(b^m)\notin[t]$, then $|\theta(b^m)|=|\pi(b^m)|$. Thus, if $q\equiv 0,1 \bmod 4$,

$$
||\theta||^2 = 1+\frac{q(q-1)}{|G|}\left(\frac{|P_1|-1}{2}\right)\left(|\pi(1)|^2-|\pi(b^m)|^2\right).
$$

If $q\equiv 3 \bmod 4$, then

$$
||\theta||^2 = 1+\frac{q(q-1)}{|G|}\left(\frac{|P_1|-1}{2}\right)\left(|\pi(1)|^2+|\pi(t)|^2-2\,|\pi(b^m)|^2\right).
$$

Observing that

$$
\frac{q(q-1)}{|G|}\left(\frac{|P_1|-1}{2}\right)=\frac{1}{(q,2)}\left(\frac{|P_1|-1}{q+1}\right),
$$

the values for $\pi\neq\psi_i$ in Table 6.3 now follow by direct substitution.

Suppose next that π is one of the characters ψ_i, and choose β according to Lemma 6.5. Summing over conjugacy classes, (6.3) becomes

$$
||\theta||^2 = 1+\frac{q(q-1)}{|G|}\left(\sum_{m=1}^{B}\left|\psi_i(b^{\beta m})\right|^2-\sum_{m=1}^{B}|\psi_i(b^m)|^2\right), \qquad (6.4)
$$

where B is the number of classes $[b^m]$. Since $|\psi_i(b^m)|^2 = 2 + \zeta^{2im} + \zeta^{-2im}$,

$$\sum_{m=1}^{B} |\psi_i(b^m)|^2 = 2B + \sum_{m=1}^{B} (\zeta^{2im} + \zeta^{-2im})$$

$$= 2B + \begin{cases} -1 & \text{if } q \equiv 0, 1 \bmod 4; \\ -2 & \text{if } q \equiv 3 \bmod 4 \end{cases} \tag{6.5}$$

by Lemma 6.2 (since the bounds on i ensure $2i \not\equiv 0 \bmod B$, and $|b|$ is even precisely when $q \equiv 3 \bmod 4$).

We consider the remaining sum in (6.4).

$$\sum_{m=1}^{B} \left|\psi_i(b^{\beta m})\right|^2 = \sum_{b^{\beta m}=1} \left|\psi_i(b^{\beta m})\right|^2 + \sum_{b^{\beta m}\in[t]} \left|\psi_i(b^{\beta m})\right|^2 + \sum_{\left|b^{\beta m}\right|>2} \left|\psi_i(b^{\beta m})\right|^2.$$

Here we have divided the classes $[b^m]$ into those in the kernel of φ, those which φ maps to an involution, and those which are mapped by φ to other classes $[b^m]$. Since $\psi_i(1) = q - 1$,

$$\sum_{b^{\beta m}=1} \left|\psi_i(b^{\beta m})\right|^2 = \left(\frac{|P_1| - 1}{2}\right)(q-1)^2. \tag{6.6}$$

Similarly,

$$\sum_{b^{\beta m}\in[t]} \left|\psi_i(b^{\beta m})\right|^2 = \begin{cases} 0 & \text{if } q \equiv 0, 1 \bmod 4; \\ 2(|P_1| - 1) & \text{if } q \equiv 3 \bmod 4 \end{cases} \tag{6.7}$$

since these classes only arise when $q \equiv 3 \bmod 4$, in which case $|\psi_i(t)| = 2$.

The sum over the remaining conjugacy classes of $\left|\psi_i(b^{\beta m})\right|^2$ can be calculated by summing $2 + \zeta^{2i\beta m} + \zeta^{-2i\beta m}$ over *all* of the classes $[b^m]$ and then subtracting that expression over the classes already considered. Thus

$$\sum_{\left|b^{\beta m}\right|>2} \left|\psi_i(b^{\beta m})\right|^2 =$$

$$\sum_{m=1}^{B}(2+\zeta^{2i\beta m}+\zeta^{-2i\beta m}) -$$

$$\left(\sum_{b^{\beta m}=1}(2+\zeta^{2i\beta m}+\zeta^{-2i\beta m}) + \sum_{b^{\beta m}\in[t]}(2+\zeta^{2i\beta m}+\zeta^{-2i\beta m})\right)$$

Observe that $\zeta^{2\beta m}=1$ whenever $b^{\beta m}=1$ or $b^{\beta m}\in[t]$. Hence by Lemma 6.2 and previous arguments,

$$\sum_{\left|b^{\beta m}\right|>2} \left|\psi_i(b^{\beta m})\right|^2 =$$

$$2B + \left\{\begin{array}{lll} -1 & \text{if } 2i\beta \not\equiv 0 \bmod |b| & \text{and } q\equiv 0,1 \bmod 4; \\ -2 & \text{if } 2i\beta \not\equiv 0 \bmod |b| & \text{and } q\equiv 3 \bmod 4; \\ 2B & \text{if } 2i\beta \equiv 0 \bmod |b|. & \end{array}\right\}$$

$$-2(|P_1|-1) - \left\{\begin{array}{ll} 0 & \text{if } q\equiv 0,1, \bmod 4; \\ 2(|P_1|-1) & \text{if } q\equiv 3 \bmod 4. \end{array}\right\} \tag{6.8}$$

Substituting from (6.5), (6.6), (6.7), and (6.8) into (6.4), (and after the obvious cancellations),

$$||\theta||^2 = 1 + \frac{q(q-1)}{|G|}\left[\left(\frac{|P_1|-1}{2}\right)(q-1)^2 \right.$$

$$+ \left\{\begin{array}{lll} -1 & \text{if } 2i\beta \not\equiv 0 \bmod |b| & \text{and } q\equiv 0,1 \bmod 4; \\ -2 & \text{if } 2i\beta \not\equiv 0 \bmod |b| & \text{and } q\equiv 3 \bmod 4; \\ 2B & \text{if } 2i\beta \equiv 0 \bmod |b| & \end{array}\right\}$$

$$-2(|P_1|-1)$$

$$\left. + \left\{\begin{array}{ll} 1 & \text{if } q\equiv 0,1 \bmod 4; \\ 2 & \text{if } q\equiv 3 \bmod 4 \end{array}\right\}\right]. \tag{6.9}$$

If $2i\beta \not\equiv 0 \bmod |b|$, then the sum of the two piecewise-defined components in (6.9) is zero. Otherwise, observing that

$$\frac{(q,2)}{2}(q+1) = \left\{ \begin{array}{ll} 2B+1 & \text{if } q \equiv 0,1 \bmod 4; \\ 2B+2 & \text{if } q \equiv 3 \bmod 4, \end{array} \right\}$$

and

$$\frac{q(q-1)}{|G|} = \frac{1}{(2,q)}\frac{2}{q+1},$$

the right hand side of (6.9) becomes

$$1 + \frac{1}{(2,q)}\frac{2}{q+1}\left[\left(\frac{|P_1|-1}{2}\right)\left((q-1)^2-4\right) + \left\{ \begin{array}{ll} 0 & \text{if } 2i\beta \not\equiv 0 \bmod |b|; \\ \frac{(q,2)}{2}(q+1) & \text{if } 2i\beta \equiv 0 \bmod |b|. \end{array} \right.\right]$$

Simplifying,

$$||\theta||^2 = 1 + \frac{1}{(2,q)}(|P_1|-1)(q-3) + \left\{ \begin{array}{ll} 0 & \text{if } 2i\beta \not\equiv 0 \bmod |b|; \\ 1 & \text{if } 2i\beta \equiv 0 \bmod |b|, \end{array} \right.$$

which is the value given in Table 6.3.

Case *(ii)* ($p = q = r$ is an odd prime):

Here q is odd, $\mathcal{N}$ consists only of the classes $[c]$ and $[d]$ (of sizes $(q^2-1)/2$), and $P_1 = P$. From (6.3),

$$\begin{aligned} ||\theta||^2 &= 1 + \frac{q^2-1}{2|G|}\left(2|\pi(1)|^2 - |\pi(c)|^2 - |\pi(d)|^2\right) \\ &= 1 + \frac{1}{q}\left(2|\pi(1)|^2 - |\pi(c)|^2 - |\pi(d)|^2\right), \end{aligned}$$

and the values in Table 6.3 follow.

Case *(iii)* (p divides $q-1$ and $q = r$ is an odd prime):

Here $\mathcal{N}$ is the set of classes $[a^l]$, each of size $q(q+1)$, along with the classes $[c]$ and $[d]$, and φ maps $(|P_1|-1)/2$ classes $[a^l]$ into the class of involutions precisely when $q \equiv 1 \bmod 4$. Recognizing that the norm of θ in case (ii) is a component of the calculation in this case, let $\left|\left|\theta_{(ii)}\right|\right|^2$ denote the case (ii) norm of θ.

If π is not one of the characters χ_j, then π is constant in absolute value across the classes $[a^l]$. Thus, from (6.3), if $q \equiv 1 \bmod 4$,

$$||\theta||^2 = \frac{q(q+1)}{|G|}\left(\frac{|P_1|-1}{2}\right)\left(|\pi(1)|^2 + |\pi(t)|^2 - 2\left|\pi(a^l)\right|^2\right) + \left|\left|\theta_{(ii)}\right|\right|^2.$$

If $q \equiv 3 \bmod 4$, then

$$||\theta||^2 = \frac{q(q+1)}{|G|}\left(\frac{|P_1|-1}{2}\right)\left(|\pi(1)|^2 - \left|\pi(a^l)\right|^2\right) + \left|\left|\theta_{(ii)}\right|\right|^2.$$

Observing that

$$\frac{q(q+1)}{|G|}\left(\frac{|P_1|-1}{2}\right) = \frac{|P_1|-1}{q-1},$$

the values for $\pi \neq \chi_j$ in Table 6.3 now follow by direct substitution.

If instead π is one of the characters χ_j, then proceeding as for ψ_i in case (i),

$$||\theta||^2 = \frac{q(q+1)}{|G|}\left(\sum_{l=1}^{A}\left|\chi_j(a^{\alpha l})\right|^2 - \sum_{l=1}^{A}\left|\chi_j(a^l)\right|^2\right) + \left|\left|\theta_{(ii)}\right|\right|^2,$$

where A is the number of classes $[a^l]$. After an analogous argument we obtain

$$\begin{aligned}
||\theta||^2 =& \frac{q(q+1)}{|G|}\Bigg[\left(\frac{|P_1|-1}{2}\right)(q+1)^2 \\
&+ \left\{\begin{array}{ll} -1 & \text{if } 2j\alpha \not\equiv 0 \bmod |a| \text{ and } q \equiv 3 \bmod 4; \\ -2 & \text{if } 2j\alpha \not\equiv 0 \bmod |a| \text{ and } q \equiv 1 \bmod 4; \\ 2A & \text{if } 2j\alpha \equiv 0 \bmod |a|. \end{array}\right\} \\
&- 2(|P_1|-1) \\
&+ \left\{\begin{array}{ll} 1 & \text{if } q \equiv 3 \bmod 4; \\ 2 & \text{if } q \equiv 1 \bmod 4 \end{array}\right\}\Bigg] + \left|\left|\theta_{(ii)}\right|\right|^2.
\end{aligned} \tag{6.10}$$

Observing that

$$\frac{q-1}{2} = \left\{\begin{array}{ll} 2A+1 & \text{if } q \equiv 3 \bmod 4; \\ 2A+2 & \text{if } q \equiv 1 \bmod 4, \end{array}\right.$$

(6.10) simplifies as before to the value given in Table 6.3. This completes the proof. □

Chapter 7

Open Questions and Future Directions

7.1 Open Questions

In Theorem 5.1 we began to determine precisely which groups possess unfaithful minimal Heilbronn characters, leaving the following question open:

Question 1. *Can we further restrict when the orthogonal and exceptional groups of Lie type possess unfaithful minimal Heilbronn characters?*

Also, although we gave a general construction in Corollary 4.6 of unfaithful minimal Heilbronn characters for those groups that possess them, not every unfaithful minimal Heilbronn character arises in this way. Foote and Wales have shown that when $p \geq 7$ is prime, $G = SL_2(p)$ always possesses a minimal Heilbronn character θ of degree (and norm) 2, the difference of characters of type χ_j of degree $p+1$ and ψ_i of degree $p-1$ [FW90, pages 23-29]. It is easy to see from Table 2.8 that this θ is unfaithful on a Sylow p-subgroup. In this case θ cannot arise in the manner of Corollary 4.6 since G has no irreducible character of degree 2.

We may ask:

Question 2. *Are there groups other than $SL_2(p)$ that possess unfaithful minimal Heilbronn characters not derived from faithful irreducible characters in the manner of Corollary 4.6, and if so is there a general characterization of such groups and a general construction for such virtual characters?*

We have shown (Lemma 1.9 part (vi)) that if G is the Galois group of a finite extension of number fields E/F, then the degree of the associated Heilbronn character is the order of zero of the Dedekind zeta function of E at the fixed point $s_0 \in \mathbb{C} - \{1\}$. Thus the degree of a minimal Heilbronn character carries important number theoretic information, and it would be significant if we could constrain the possibilities:

Question 3. *Can we compute the minimal degrees of unfaithful minimal Heilbronn characters for each family of quasisimple groups?*

Note that Question 3 is related to Question 2 in that, if the only unfaithful minimal Heilbronn characters for a particular family of quasisimple groups are those arising in the manner of Corollary 4.6, then the minimal degree of an unfaithful minimal Heilbronn character of a group in that family is simply the minimal degree of a faithful irreducible character.

Another outstanding quantitative question is:

Question 4. *What is the smallest possible norm for an unfaithful minimal Heilbronn character of a given group G?*

We turn to a question of a more abstract nature. Consider a directed graph whose vertices correspond to the conjugacy classes of a given group G. To simplify the discussion we will use the terms "vertex" and "conjugacy class" interchangeably, so that it makes sense to refer to the "elements" of a given vertex. Choose an initial vertex α, in particular one corresponding to a class of p-elements for some prime p, and draw an arrow to each vertex β having an element contained in a proper subgroup of G generated by elements of α. Continue in this manner, extending arrows from a given vertex β to each vertex having an element in a proper subgroup generated by elements of β together with elements of any of its predecessors.

Now suppose θ is a minimal Heilbronn character of G and θ is constant on the conjugacy class corresponding to the initial vertex α (i.e. that class is in the kernel of the restriction of θ to an appropriate proper subgroup). Then by Lemma 3.3, θ is constant as well on the successors of α. If every vertex is a successor of α, then θ is constant on all of G, a contradiction (since in that case θ is a multiple of the principal character of G, hence a character and not a minimal Heilbronn character of G). In fact if any vertex corresponding to a class of involutions is a successor of α, a contradiction of Theorem 2.1

arises. Thus if G possesses an unfaithful minimal Heilbronn character, then for some class α of p-elements the directed graph with initial vertex α does not span all of G, and in particular includes no involutions.

As every group G has a well-defined, fixed set of such directed graphs (one per conjugacy class), the question arises:

Question 5. *Can the proof of Theorem 1 be rephrased in terms of directed graphs* (G, α)*, and are these directed graphs of independent interest?*

7.2 Future Directions

The interplay between strong closure and the characters of a finite group plays a fundamental role in the development of our main theorem. Since the strongly closed subgroups we consider are associated with a fixed prime p, a natural sequel to this investigation is the consideration of the implications of strong closure on p-modular character theory. This leads to questions under the rubric of *fusion systems*, a category introduced by L. Puig to generalize fusion phenomena in finite groups and their p-blocks. Michael Aschbacher has outlined a strategy relating the classification of simple saturated 2-fusion systems to the Classification of Finite Simple Groups (and thus to the ongoing second generation proof — the "Revision" — of the Classification of Finite Simple Groups) [Asc09]. The strategy is based on the observation that many important theorems are more easily proven for fusion systems than for groups, and that corresponding theorems for finite groups can often be coaxed from their fusion system analogues. There are many remaining open questions in this program that would be reasonable sequels to the work in this dissertation. For example:

Problem. *Determine all possible simple fusion systems* $\mathcal{F}$ *based on a finite 2-group* S *and such that* $C_{\mathcal{F}}(t)$ *has a component of alternating type for some fully centralized involution* $t \in S$ *such that* $C_S(t)$ *contains Baum*(S)*, the Baumann subgroup of* S*.*

This addresses one of the many special cases of the proposed classification, and there are analogous problems in which the centralizer of an involution has a component of type $L_2(q)$, ${}^2G_2(q)$, a sporadic simple group, or a group of Lie type in characteristic 2. A goal for the investigation of this problem would be the discovery of techniques that could be brought to bear on some or all of these cases.

The combination of strong closure and modular character theory leads as well to more classically motivated problems, also with application to the Revision. Consider D. Goldschmidt's theorem [Gol75]:

Theorem. (Goldschmidt) *Suppose the finite group G contains a direct product of two strongly closed 2-subgroups, $S_1 \times S_2$. Then $[\langle {S_1}^G \rangle, \langle {S_2}^G \rangle] \leq O_{2'}(G)$.*

(Here the connection to modular character theory becomes evident after recalling that, for any prime p, $O_p(G)$ is contained in the intersection of the kernels of the p-modular irreducible representations of G [Isa76, p. 280].) As a corollary, Goldschmidt showed that $\langle {S_1}^G \rangle \cap \langle {S_2}^G \rangle \leq O_{2'}(G)$ whenever $S_1 \times S_2$ is Sylow in G. In this case, assuming that both S_1 and S_2 are nontrivial, it follows that $\langle {S_1}^G \rangle$ and $\langle {S_2}^G \rangle$ are *proper* normal subgroups of G, hence G is not simple.

A partial generalization of Goldschmidt's theorem to odd primes that would be applicable to the Revision is:

Problem. *Let G be a finite group with an abelian Sylow p-subgroup A, and suppose that B is a strongly closed subgroup of A with respect to G. Then G has a normal subgroup N having a Sylow p-subgroup B_1 such that $\Omega_1(B_1) = \Omega_1(B)$. In particular, if A is elementary abelian, then $B \in Syl_p(N)$.*

The implication for simple groups G is that B must contain all of $\Omega_1(A)$ (in which case the statement is satisfied vacuously by $N = G$).

A special case of this problem of independent interest is when A is an abelian Sylow 3-subgroup of rank 3 and $B \cong Z_3$. For a reasonable test case we may assume further that $C_G(B)/O_{3'}(C_G(B)) = B \times L$ with $L \cong A_7$ or S_7. Another related test case of small rank is:

Problem. *Let G be a finite group with a Sylow 3-subgroup P of order 9 such that $N_G(P)/O_{3'}(C_G(P)) \cong S_3 \times S_3$. Then G has a normal subgroup N with $|N|_3 = 3$.*

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