FUSION SYSTEMS WITH STANDARD COMPONENTS OF SMALL RANK

A Dissertation Presented

by

Matthew Welz

to

The Faculty of the Graduate College

of

The University of Vermont

In Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
Specializing in Mathematics

October, 2012

Accepted by the Faculty of the Graduate College, The University of Vermont, in partial fulfillment of the requirements for the degree of Doctor of Philosophy, specializing in Mathematics.

Dissertation Examination Committee:

______________________________ Advisor
Richard Foote, Ph.D.

____
John Voight, Ph.D.

____
Greg Warrington, Ph.D.

____
J. Michael Wilson, Ph.D.

______________________________ Chairperson
Eric Hernandez, Ph.D.

______________________________ Dean, Graduate College
Domenico Grasso, Ph.D.

Date: August 2, 2012

Abstract

In this thesis we study two problems in the area of fusion systems which are designed to mimic, simplify, and generalize parts of the Classification of Finite Simple Groups. In general, a finite simple group G is determined to a great extent by the structure and conjugacy pattern of a Sylow 2-subgroup. A 2-fusion system considers only a 2-group S equipped with a family of injective homomorphisms (called fusion maps) on subgroups of S without reference to an ambient group G. The general framework of fusion systems also arises naturally in the study of modular representations and classifying spaces; and so results proved for fusion systems have potential ramifications beyond the realm of finite group theory. One problem in this area is to determine S or, whenever possible, the entire 2-fusion system only from the knowledge of certain subgroups and fusion maps between these subgroups. In this thesis we consider two such problems: where S contains subgroups and fusion maps that arise in the Classification with standard components of type $SL_2(q)$ and $PSL_2(q)$. In particular, we give a characterization of simple, saturated fusion systems containing such components.

Acknowledgments

I owe my deepest gratitude to my advisor, Richard Foote. For six years I have been the beneficiary of his wisdom, enthusiasm, and advocacy. His breadth of knowledge and expertise are only matched by his amazing gift of exposition; Dr. Foote has inspired me both as a mathematician and a teacher. It has been a great privilege to spend many a morning in his office discussing both the heady and mundane; whether normal p-complements or lawn mower maintenance, I always left his office edified and encouraged. Thank you, Dr. Foote.

I am grateful to Dr. John Voight, Dr. Greg Warrington, Dr. J. Michael Wilson, and Dr. Eric Hernandez for serving on my Dissertation Examination Committee. I appreciate your time, patience, and thoughtful criticisms and suggestions.

To all my math friends, but in particular to Nick Teff, Hy Ginsberg, and Justin Lynd, thank you. Your friendship has benefited me both mathematically and personally.

Miss Jones, one of my earliest (and most dogged!) advocates. Thank you for your friendship, abiding belief, and occasional friendly nagging through all my ups, downs, and general hijinks.

Kirsten, a kind and supportive partner. Thank you for your honesty, sympathy, and sacrifice; there is no way this thesis gets finished without you.

Finally, to Soren, thank you for all the ~~sleepless~~ restful nights this past year. No matter how long or trying the day, it was with the greatest anticipation and joy that I came home to see you each evening; you are a wonderful little man.

Table of Contents

Acknowledgments ii

1 Introduction **1**

1.1 Group Theory Background 4

1.2 Standard Components 5

1.3 Standard Form Problems for $SL_2(q)$ and $L_2(q)$ 7

1.3.1 Standard Components of $SL_2(q)$-type 8

1.3.2 Standard Components of $L_2(q)$-type 10

2 Fusion Systems and Results **14**

2.1 Introduction to Fusion Systems 14

2.2 Saturated Fusion Systems 17

2.3 Quotients and Normal Subsystems 22

2.4 Simple Fusion Systems 26

2.5 The Main Results of the Thesis 29

2.5.1 The $SL_2(q)$ Standard Form Problem 29

2.5.2 The $L_2(q)$ Standard Form Problem 31

3 Preliminary Group Theory Lemmas **35**

3.1 The Automorphism Groups of D_{2^n} and Q_{2^n} 35

3.2 Notes on Some 2-groups 39

3.3 Miscellaneous Lemmas 42

3.4 Facts about $SL_2(q)$ and $L_2(q)$ 45

4 **Transfer and Preliminary Fusion Systems Lemmas** **47**

4.1 Transfer 47

5 **The Group Theoretic Proof for $SL_2(q)$** **57**

6 **The Proof of Theorem 1** **72**

7 **The Group-Theoretic Proof for $L_2(q)$** **86**

7.1 The $|T|$=2 case 88

7.2 The $|T| > 2$ case 91

8 **Proof of Theorem 2** **98**

8.1 The $|T| = 2$ case 103

8.2 The $|T| > 2$ case 106

9 **Future Work** **116**

9.1 Further Research I 116

9.2 Further Research II 117

9.3 Further Research III 118

9.4 Further Research IV 119

Bibliography 119

Chapter 1

Introduction

If G is a group [1] then for any subgroup H of G and any subsets X, Y of H we say X and Y are *fused* in G if $X^g = Y$ for some $g \in G$, i.e. X and Y are conjugate in the overgroup G (and are conjugate in H itself if g can be chosen from H; so H-conjugate subsets are necessarily G-fused in H too). Alperin's Fusion Theorem tells us that if X, Y are fused in a Sylow p-subgroup $S \leq G$ by g then the action of g can be achieved by a sequence of conjugations in normalizers of subgroups of S, that is, fusion is achieved *locally*. Analysis of fusion in G yields many important results in group theory and has played a prominent role in the Classification of Finite Simple Groups in the last century.

Fusion systems are a construction that considers a group H together with a compatible set of homomorphisms between subgroups of H that generalizes the set of all (fusion) maps induced by fusion in G as above when H occurs as a subgroup of some G. In other words, the "potential overgroup" G is replaced by a set of maps – and in this abstract, axiomatic setting there may not even be a group G that "realizes" these maps (we will say more about this in the next chapter).

The notion of a fusion system is originally due to Lluis Puig who encoded fusion data of p-groups into what he called a *Frobenius Category* [Pui06] (which we now call a saturated fusion system). Puig used fusion systems as a tool in his study of modular representation theory and p-blocks of finite groups. Since then, homotopy theorists

[1] Unless specifically mentioned otherwise, groups throughout the thesis will be assumed to be finite. Basic results and notation may be found in [Asc00] or [Gor80].

(see for example [BLO03]) have taken up this theory and pushed it forward in their study of p-completed classifying spaces of finite groups. Finite group theorists have taken an interest in fusion systems as some results on groups might be more easily proved in the category of saturated fusion systems; we pick up here.

Since the advent of fusion systems, some parts of the Classification of Finite Simple Groups might be improved by first proving results for fusion systems and then applying those results to groups. For example, take a finite group G with $S \in Syl_2(G)$ and let $\overline{G} = G/O_{2'}(G)$ where $O_{2'}(G)$ is the maximal normal odd-order subgroup of G. It turns out that G and $\overline{G} = G/O_{2'}(G)$ induce exactly the same fusion pattern in S (this actually holds for any prime p). So "cores" (i.e. $O_{2'}(G)$) essentially disappear in the study of fusion systems. A significant amount of effort in the original Classification is expended on dealing with cores in various subgroups of simple groups, so fusion systems offer a potential for avoiding (or at least "postponing dealing with") such problems.

In the Classification, the simple groups are split between those of *characteristic 2-type* and those of *component type*. Following a major program of research Aschbacher laid out in [AKO11], we work toward a classification of fusion systems of "component type" in order to establish a new proof of the Classification for groups of component type. To that end, this thesis describes the classification of the simple, saturated 2-fusion systems in the following cases:

(1) A "small case" of the Classical Involution Theorem [Asc77a, Asc77b] by considering fusion systems $\mathcal{F}$ over a finite 2-group S possessing a weakly closed (generalized) quaternion subgroup R which is also strongly closed in the centralizer of its unique involution. We will assume further that $Q = C_S(R)$ is "tightly embedded" in $\mathcal{F}$. This is the fusion system version of a standard component of type $SL_2(q)$, q odd.

(2) A fusion system version of the Standard Form problem for $L_2(q)$, q odd, in which the centralizer of our standard component $L_2(q)$ has 2-rank bigger than 1.

In this first chapter we introduce the basic concepts from group theory and discuss briefly the Classification of Finite Simple Groups.

In Chapter 2 we provide a basic introduction to the theory of fusion systems. We give particular attention to the similarities and differences between group fusion and fusion systems in general, consider several examples, and establish Alperin's Fusion Theorem; we follow the more thorough treatments given in [AKO11] and [Cra11b]. In the final section of the chapter, after the background and motivation from group theory and fusion systems have been laid out, we give the precise statements of the main results of the thesis.

In Chapter 3 we collect a number of preliminary group theory lemmas. In particular, we establish several results on various 2-groups that play prominent roles in our analysis. We also include some basic and important results on $L_2(q)$ and $SL_2(q)$. In Chapter 4 we collect a few preliminary results on fusion systems. We discuss transfer in groups and fusion systems, and present Justin Lynd's proof of a fusion systems version of Thompson's Transfer Lemma. This lemma is essential to proving our first main result, Theorem 1.

The main theorems are proved in Chapters 6 and 8. In both cases, we establish the result first in group-theoretic terms and then mimic these methods whenever possible to establish the result for fusion systems in general; these group-theoretic "templates" occupy Chapters 5 and 7 respectively. The group-theoretic results are already known but were originally proved under different hypotheses with different methods. In particular, the original proofs were not strictly fusion-theoretic. Finally, we conclude with a discussion of future work and new directions in Chapter 9.

1.1 Group Theory Background

For the convenience of the reader, we will review a selection of particularly relevant concepts and definitions. Throughout the thesis G is a finite group and p is a prime.

We say a group L is *perfect* if $L = L'$, and say L is *quasisimple* if L is perfect with $L/Z(L)$ a nonabelian simple group. The *components* of G are its subnormal, quasisimple subgroups. A subgroup L of G is a *2-component* if L is perfect and $L/O_{2'}(L)$ is quasisimple, where $O_{2'}(L)$ is the core of L, i.e., the largest normal subgroup of L of odd order. It turns out that distinct components of G commute, and we denote the central product of all components in G by $E(G)$. The *Fitting* subgroup, $F(G)$, is the maximal nilpotent normal subgroup of G. It turns out that $F(G)$ is the direct product of $O_p(G)$ for all primes p dividing $|G|$, where $O_p(G)$ is the largest normal subgroup of p-power order in G. We define the *generalized Fitting* subgroup of G to be $F^*(G) = F(G)E(G)$.

We introduce some notation and definitions for p-groups. Let P be a p-group. Then let $\Omega_1(P)$ denote the characteristic subgroup of P generated by all elements of order p. If P is abelian then $\Omega_1(P)$ is an elementary abelian p-group. Another important characteristic subgroup is the *Frattini* subgroup, $\Phi(P)$, which is the intersection of all maximal subgroups in P. It follows that $P/\Phi(P)$ is elementary abelian.

The Classification of Finite Simple Groups was an immense undertaking, filling up thousands of journals pages by scores of authors. This Classification was the driving force behind much of the development of modern finite group theory and largely motivates this thesis as well. For that reason, we give a *rough* outline of the Classification and how it ties in with our work.

We first discuss the partition of simple groups into those of characteristic 2-type and component type. A group G is of *component type* if $C_G(t)$ has a 2-component for some involution $t \in G$; whereas a group is of *characteristic 2-type* if $F^*(C_G(t)) = O_2(C_G(t))$ for all involutions $t \in G$. The *B-conjecture* asserts that 2-components in $C_G(t)$ are quasisimple (i.e., are components), for all involutions t in G (in a simple group of component type). The B-conjecture was ultimately proved in the course of the Classification but before that it was imposed as an “axiom” in order to effect the dichotomy: every simple group is either of component type or characteristic 2-type.

1.2 Standard Components

The simple groups consist of the cyclic groups of prime order, alternating groups of degree 5 or greater,the 16 infinite families of simple groups of Lie type, and 26 sporadic groups. As a consequence of the dichotomy theorem the Classification breaks into very distinct “halves” which necessitated the development of different methods and techniques. It is the finite simple groups of *component* type with which this thesis is concerned.

Let $GL_n(q)$ be the general linear group of $n \times n$ invertible matrices over the field of q elements. By $SL_n(q)$ we mean the subgroup of determinant 1 matrices of $GL_n(q)$. Take the quotient of $SL_n(q)$ by its scalar matrices; this quotient is the projective special linear group $PSL_n(q)$ which we usually write as $L_n(q)$. Except for a few small cases, $SL_n(q)$ is quasisimple and $L_n(q)$ is simple. Of particular importance to this thesis is the fact that if $q > 3$ then $SL_2(q)$ is quasisimple and $L_2(q)$ is simple.

In light of the conclusion of the Classification – that “most” simple groups are of Lie type – we can think of a generic simple group G as $GL_n(q)$ (it is easier to illustrate our concepts in the “universal” group $GL_n(q)$ rather than in its simple section $L_n(q)$). Then G is of characteristic 2 or component type according to whether q is even or odd respectively (again, except for some small cases). This difference manifests itself in the fact that an involution of G is diagonalizable if and only if the field q is odd. In this odd case, let $t \in G$ be a diagonal matrix with k eigenvalues -1 (and $n-k$ eigenvalues 1). Then it follows that $C_G(t) \cong GL_k(q) \times GL_{n-k}(q)$ where the direct factors of this decomposition are subgroups of block matrices acting on the eigenspaces of t. Then, roughly speaking, the subgroups $GL_k(q)$ and $GL_{n-k}(q)$ are the components of $C_G(t)$. Among these centralizers of involutions there is, loosely speaking, a "largest" component, L, namely this happens when $k = n-1$ (or $k = 1$); here $L \cong GL_{n-1}(q)$ and $C_G(L)$ is the cyclic group $GL_1(q)$ (which is not a component). We use these "largest" components L as templates for defining "standard components" with their concomitant properties (as established in Aschbacher's Standard Form Theorem). As this thesis will be concerned with components of type $SL_2(q)$ and $L_2(q)$, one might say we are considering the case that $n = 3$ above.

Aschbacher showed in the Classification — assuming the B-conjecture holds — that (simple) groups of component type always possess standard components (with a few small, classifiable exceptions). The Classification then proceeded by taking all known possibilities for L as a standard component in some arbitrary (unknown) simple group G, and proved in each case that G is in fact known too, i.e., solving all "standard form" problems. This essentially gave an inductive classification of all component type groups (again, assuming the B-conjecture was true).

We now give a precise definition of standard components. A subgroup C of an arbitrary group G is *tightly embedded* if C has even order and $|C \cap C^g|$ has odd order for every $g \in G - N_G(C)$. A component L of $C_G(t)$ for some involution $t \in G$ is *standard* in G if the following are satisfied:

(i) $L \trianglelefteq C_G(t)$,

(ii) $[L, L^g] \neq 1$ for all $g \in G$,

(iii) the subgroup $C = C_G(L)$ is tightly embedded in G, and

(iv) $N_G(L) = N_G(C)$.

Notice that we say G has a standard component L even though L is *not* a component of G itself, but rather L is a component of $C_G(t)$, for some involution $t \in G$.

As this thesis is concerned with standard components (albeit in fusion systems), we present another especially relevant example here.

Example 1.1. Let G be the alternating group A_{10}, let $t = (1\ 2)(3\ 4)$, $u = (1\ 2)(5\ 6)$, and $C = A_{\{1,2,3,4\}} \cong A_4$. Let $W = \langle\, (1\ 2)(3\ 4), (1\ 3)(2\ 4)\,\rangle$, so $W = C'$. Then it follows that

$$C_G(t) = (W \times L)\langle\, u\,\rangle \text{ where } L = A_{\{5,6,7,8,9,10\}} \cong A_6.$$

Observe that u normalizes both C and L with $W\langle\, u\,\rangle \cong D_8$, $C\langle\, u\,\rangle \cong S_4$ and $L\langle\, u\,\rangle \cong S_6$. As A_6 is simple, it is immediate that L is a (quasisimple) component of $C_G(t)$ such that $[L, L^g] \neq 1$ for any $g \in G$. It is easy to check that $C_G(L) = C$ and $N_G(L) = (C \times L)\langle\, u\,\rangle$ so that $N_G(C_G(L)) = N_G(L)$. Notice that

$$W \cap W^g \neq 1 \text{ if and only if } C^g = C$$

so C is tightly embedded, and thus L is a standard component.

An important property of standard components — which is an easy consequence of (i)-(iv) above and which we will use repeatedly without mention — is that $L \trianglelefteq C_G(t_1)$ for *any* involution $t_1 \in C_G(L)$.

1.3 Standard Form Problems for $SL_2(q)$ and $L_2(q)$

Of all the possibilities for a standard component, the $SL_2(q)$ and $L_2(q)$ for q odd cases (with cores) were both particularly difficult and fundamental to the Classification. For example, the standard form problem for $SL_2(q)$ is a bedrock case in Aschbacher's seminal Classical Involution paper [Asc77a, Asc77b]. It is the purpose of this thesis to describe fusion systems with components of type $SL_2(q)$ or $L_2(q)$ so we discuss these standard form problems in greater detail. First, we are fortunate that in the setting of fusion systems, cores contribute nothing to the fusion pattern, so our work is simplified at the outset. A principal issue in dealing with the fusion systems version of these problems is what the definition of standard should include. We conclude this chapter by discussing each case individually in group-theoretic terms before completing the discussion and providing precise fusion systems statements in Section 2.5.

1.3.1 Standard Components of $SL_2(q)$-type

As the definitions of centralizers and normal subsystems are somewhat elusive in the context of fusion systems (this is discussed in greater detail in Section 2.4), we do our best to strip them away from our working definition of "standard". In the case of $SL_2(q)$ we can do this nicely. Suppose G is an arbitrary finite simple group with an involution t such that $C_G(t)$ has a standard component L of type $SL_2(q)$ for q odd. Set $C = C_G(L)$ and $N = N_G(L)$. Let $S \in Syl_2(G)$ be chosen such that $t \in S$ and $S \cap N \in Syl_2(N)$. Let $R = S \cap L \in Syl_2(L)$. By Lemma 3.16, R is a generalized quaternion group. As L is standard, we have that

(i) $L \trianglelefteq C_G(t)$ and $N_G(C) = N$,

(ii) $[L, L^g] \neq 1$ for all $g \in G$, and

(iii) $C_G(L)$ is tightly embedded in G.

We claim that in this situation the following two conditions hold:

(S1) R is *weakly closed* in G, i.e., whenever $R^g \leq S$ for some $g \in G$ then $R^g = R$, and

(S2) R is *strongly closed* in $C_G(y)$ for any involution $y \in C_S(R)$, i.e., if $r \in R$ and $\langle R, r^h \rangle$ is a 2-group for any $h \in C_G(y)$, then $r^h \in R$.

We now show that L being standard implies that the conditions (S1) and (S2) hold.

Lemma 1.2. *Let G be a finite group with a standard component L of type $SL_2(q)$ for q odd. If $R \in Syl_2(L)$ then R is weakly closed in G, and R is strongly closed in $C_G(y)$ for any involution $y \in C_S(R)$.*

Proof. To see that R is weakly closed in S, by Lemma 3.3 take a conjugate $R^g \leq N_G(R)$ for some $g \in G$ so that

$$[R, R^g] \leq R \cap R^g.$$

Let z be the unique involution of R and recall that $\langle z \rangle = Z(L) \leq C$ (here z is the negative of the 2×2 identity matrix in $L = SL_2(q)$). If $z \in R \cap R^g$ then since C is tightly embedded, we get that $g \in N$. Since $\langle R, R^g \rangle$ is a 2-group contained in L and R is Sylow in L, it follows that $R^g = R$. Suppose now that $R^g \cap R = 1$ so that $[R, R^g] = 1$. Since R centralizes z^g and R^g centralizes z, tight embedding gives us that R normalizes L^g and R^g normalizes L. By Lemma 3.16 no nontrivial 2-power automorphisms of L centralize R, so $R \leq C_G(L^g)$ and $R^g \leq C_G(L)$. Tight embedding now forces L^g and L to normalize each other so that L and L^g are components of LL^g. Distinct components commute by [Asc00, 31.5], so our standard form hypothesis forces $L = L^g$, i.e. $g \in N_G(L)$. As $\langle R, R^g \rangle$ is a 2-group contained in L we again get $R^g = R$. Thus R is weakly closed in G. In particular, $R \trianglelefteq S$, i.e. $S \leq N$.

Finally, as above, for every involution $y \in C_S(R) = C_S(L)$, tight embedding forces $C_G(y) \leq N$. Thus R is a Sylow 2-subgroup of the normal subgroup L of $C_G(y)$, hence is easily seen to be strongly closed in $C_G(y)$. $\square$

This argument has shown that the group-theoretic axioms for a standard component of type $SL_2(q)$ imply the "fusion-theoretic" properties of its Sylow 2-subgroup R. Thus we will focus on solving the (stronger) "standard form" problem for fusion systems, only assuming (S1) and (S2). This will constitute our Theorem 1 which, for the sake of completeness, we state here.

Theorem 1. Let $\mathcal{F}$ be a simple saturated fusion system on a 2-group S with R a generalized quaternion subgroup of S. Assume further that

(1) R is weakly closed in S with respect to $\mathcal{F}$, and

(2) R is strongly closed in $C_{\mathcal{F}}(y)$ for all involutions $y \in C_S(R)$.

Then either $\mathcal{F}$ is the fusion system of $L_3(q)$ or $G_2(q)$ for some odd q, or S contains a strongly $\mathcal{F}$-closed quasidihedral subgroup.

1.3.2 Standard Components of $L_2(q)$-type

We suppose now that G is a simple group with involution $t \in G$ such that L is a standard component isomorphic to $L_2(q)$ for some odd $q > 3$, and let $Q \in Syl_2(C_G(L))$. We consider the particular case where $m(Q) \geq 2$. This scenario is very different from the $m(Q) = 1$ case already treated in [Lyn12] in which Q plays little role. When $m(Q) \geq 2$ at the outset we know very little about what Q could be. One example where $m(Q) = 2$ was illustrated in the above example when $G = A_{10}$ because $L \cong A_6 \cong L_2(9)$; in this example $Q = W$ is a four-group.

To get a handle on Q in general we consider fusion of four-groups contained in Q. One of the main properties used in the original $L_2(q)$ standard form problem is that if $q > 9$ and U is any four-group contained in $N = N_G(L)$ then

$$L \leq \Gamma_{U,1}(L) \qquad \text{where} \qquad \Gamma_{U,1}(L) = \langle\, C_L(u) \mid u \in U^\sharp \,\rangle, \tag{1.1}$$

by Lemma 3.19. The importance of this "Γ_1-generation" property of $L_2(q)$ is that it imposes restrictions on the fusion of four-subgroups of $C_G(L)$: More precisely, it gives that if W is any four-group in $C_G(L)$ and $U = W^g \leq N_G(L)$ for any $g \in G$, then by tight embedding we get

$$L = \Gamma_{U,1}(L) = \langle\, C_L(u) \mid u \in W^{g\sharp} \,\rangle \leq N_G(L^g) = N^g.$$

It then follows easily (the details are given in Chapter 7) that some four-group of $C_G(L)$ is contained in N^g, so symmetrically $L^g \leq N$. As before, since L and L^g normalize each other and L is standard, $L = L^g$. This argument shows — under the Γ_1-generation property (1.1) — that if W is any four-group in $C_G(L)$ and $W^g \leq N_G(L)$ for any $g \in G$, then $W^g \leq C_G(L)$. In our fusion systems standard form problem we impose a weaker form of this fusion: if $W \leq C_G(L)$ is a four-group such that $W^g \leq N_G(L)$ for any $g \in G$, then $W^g \cap C_G(L) \neq 1$. We refer to this as *the* Γ_1-*property of* $L_2(q)$ *standard components.* We reiterate that this property *does* hold in the group-theoretic standard form situation for all odd $q > 9$.

We now state our second main theorem and refer the reader to Section 2.5 for a more thorough discussion of the hypotheses.

Theorem 2. Let $\mathcal{F}$ be a simple saturated fusion system on a 2-group S. Suppose that $\mathcal{F}$ contains a nontrivial subgroup Q and a subsystem $\mathcal{K}$ on $R \leq S$ of type $L_2(q)$, with q odd, such that $|R| = 2^m$ and $m \geq 3$. Suppose further that

(1) $\mathcal{K}$ is a normal component of $C_{\mathcal{F}}(z)$ for every fully $\mathcal{F}$-centralized involution $z \in Z(Q)$,

(2) $C_{\mathcal{C}}(\mathcal{K})$ is a fusion system on Q, where z is any involution as in (1) with $\mathcal{C} = C_{\mathcal{F}}(z)$, and

(3) $C_{\mathcal{C}}(\mathcal{K})$ is Γ_1-embedded for some z as in (1).

Then if Q has rank at least 2 then one of the following holds:

(i) Q is $\mathcal{F}$-strongly closed, or

(ii) Q is a dihedral subgroup of index 2 contained in a strongly closed dihedral group.

For the sake of completeness we end this section by showing why the Γ_1-property fails in the group-theoretic standard form problem for $L = L_2(9)(\cong A_6)$, again referring to Example 1.1 when $G = A_{10}$.

Example 1.3. There exists a four-group U such that $\Gamma_{U,1}(L) < L$ in A_{10}, which is essentially caused by the exceptional isomorphism $L_2(9) \cong A_6$ which has Sylow 2-subgroups isomorphic to D_8. To see why this exceptional isomorphism causes the Γ_1-property to fail for $q \leq 9$ (only) let $t = (1\ 2)(3\ 4)$ in $G = A_{10}$ and let $W = \langle\, (1\ 2)(3\ 4), (1\ 3)(2\ 4)\,\rangle$. Then let $U = W^g = \langle\, (5\ 6)(7\ 8), (5\ 7)(6\ 8)\,\rangle$ where $g = (1\ 5)(2\ 6)(3\ 7)(4\ 8)$. Again, let $L = A_{\{5,6,7,8,9,10\}}$ so $L \cong A_6$. Then $U \leq L$ (and hence normalizes L) and one easily computes that

$$\Gamma_{U,1}(L) = A_{\{5,6,7,8\}} \cdot \langle\, (5\ 6)(9\ 10)\,\rangle \cong S_4.$$

Thus $\Gamma_{U,1}(L) < L$, i.e. *the* Γ_1*-property fails for some four-group* $U = W^g$ *such that* $W^g \leq N_G(L)$ *and* $W^g \cap C_G(L) = 1$.

Indeed $G = A_{10}$ and A_{11} are the *only* simple groups with standard components L that have D_8 Sylow 2-subgroups and have the 2-rank of $C_G(L) \geq 2$ (here $L \cong L_2(9) \cong A_6$ and $L \cong A_7$ respectively, and these two components have the same 2-fusion patterns) – see [Foo78]. Thus there is hope that this "small" case of the $L_2(q)$

fusion system standard form problem – without the Γ_1-property assumption – can also be resolved (see Chapter 9).

Chapter 2

Fusion Systems and Results

2.1 Introduction to Fusion Systems

In this introduction to fusion systems we follow a mix of [AKO11] and [Cra11b]. Because this is a new and emerging field, this chapter includes a sufficient introduction to follow this thesis but is by no means intended to be exhaustive. First we introduce the terminology and notation necessary to define an abstract fusion system. Let G be a group. For $g \in G$, let c_g denote the automorphism of G given by $x \mapsto= x^g = g^{-1}xg$. If P and Q are subgroups of G, we define

$$\mathrm{Hom}_G(P, Q) = \{c_g \mid g \in G \text{ and } P^g \leq Q\}.$$

If $Q = P$ in the above definition then

$$\mathrm{Hom}_G(P, P) = \mathrm{Aut}_G(P) \cong N_G(P)/C_G(P) \leq \mathrm{Aut}(P).$$

Let $\mathrm{Inj}(P, Q)$ be the set of all injective homomorphisms from P into Q.

We consider the motivating example for what we will shortly define as a fusion system. Let $S \in Syl_p(G)$ and let $\mathcal{F}_S(G)$ denote the category whose objects are the subgroups of S and whose morphisms are

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q) \text{ for } P, Q \leq S.$$

Many definitions and results on group fusion can now be stated in terms of this category. For example, a subgroup of $H \leq G$ containing S is said to *control fusion* in S if and only if $\mathcal{F}_S(G) = \mathcal{F}_S(H)$. Then a theorem of Burnside [Gor80, 7.1.1] tells us that if S is abelian then $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$, i.e, two elements of S are conjugate in G if and only if they are conjugate in $N_G(S)$.

We now define an abstract fusion system as in [BLO03]; this a refinement of Puig's original definition [Pui06].

Definition 2.1. A *fusion system* is a category $\mathcal{F}$ over a p-group S whose objects are the subgroups of S, and whose morphisms, $\mathrm{Mor}_{\mathcal{F}}(P, Q)$, satisfy the following axioms for all $P, Q \leq S$:

(i) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$; and

(ii) for all $\phi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$, the isomorphism $\phi : P \to \phi(P)$ belongs to $\mathrm{Mor}_{\mathcal{F}}(P, \phi(P))$ and its inverse ϕ^{-1} belongs to $\mathrm{Mor}_{\mathcal{F}}(\phi(P), P)$.

Following [AKO11] we will write $\mathrm{Mor}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Function composition $\phi\psi$, whenever it is defined, means "first apply ϕ, then ψ". If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $g \in P$ we will usually apply ϕ to g as a superscript, i.e. g^{ϕ}, to emphasize the analogy with group fusion. Likewise, if $\psi \in \mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Mor}_{\mathcal{F}}(P, P)$ we will denote the map $\phi^{-1}\psi\phi \in \mathrm{Aut}_{\mathcal{F}}(P^{\phi}, P^{\phi})$ by ψ^{ϕ}. If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ and $s \in S$, then $(P^{\phi})^s = P^{\phi c_s}$, i.e. "$\phi s$" is also a fusion map. When $P \cong Q$ we denote the set of all isomorphisms in $\mathcal{F}$ from P into Q by $\mathrm{Iso}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

Axiom (i) guarantees that any fusion system $\mathcal{F}$ contains $\mathcal{F}_S(S)$, i.e., $\mathcal{F}$ contains all conjugations induced by elements of S. In particular, all inclusions $P \hookrightarrow Q$ are morphisms (conjugation by the identity restricted to P). Since all maps in $\mathcal{F}$ are injective, we observe that $\mathrm{Hom}_{\mathcal{F}}(P, P) = \mathrm{Aut}_{\mathcal{F}}(P) \leq \mathrm{Aut}(P)$ for $P \leq S$. Finally, we define $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\,\mathrm{Inn}(P)$ to be the outer automorphisms of P in $\mathcal{F}$.

We observe that the category $\mathcal{F}_S(G)$ is a fusion system. It follows that there is a largest fusion system $\mathcal{U}(S)$ on S given by

$$\text{Hom}_{\mathcal{U}(S)}(P, Q) = \text{Inj}(P, Q) \text{ for all } P, Q \leq S.$$

On the other hand, $\mathcal{F}_S(S)$ is the smallest fusion system on S. As it stands, the definition of an abstract fusion system is too general for most purposes. For example, if S is the Klein four-group then $\text{Aut}_{\mathcal{U}(S)}(S) = \text{Aut}(S) \cong S_3$ which has Sylow 2-subgroups isomorphic to Z_2. But in this case $\text{Inn}(S)$ is trivial so S is not "Sylow" in $\mathcal{F}$. Also, the fusion in an abstract fusion system may not take place "locally". For those reasons (among others), we define *saturated* fusion systems, in short, a fusion system $\mathcal{F}$ is saturated if it admits a form of Alperin's Fusion Theorem and in which S is a "Sylow" subgroup of $\mathcal{F}$. First we shall need some definitions.

Let $\mathcal{F}$ be a fusion system on a p-group S. Two subgroups $P, Q \leq S$ are said to be *$\mathcal{F}$-conjugate* if $P^\phi = Q$ for some $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$. Let $P^{\mathcal{F}}$ be the set of all $\mathcal{F}$-conjugates of P, that is, the orbit of P under the action of $\mathcal{F}$. Likewise for an element $u \in S$. the $\mathcal{F}$-conjugates of u, denoted by $u^{\mathcal{F}}$, are the elements u^ϕ for $\phi \in \text{Hom}_{\mathcal{F}}(\langle u \rangle, S)$.

Definition 2.2. Let $\mathcal{F}$ be a fusion system over a p-group S with $P \leq S$. Then

- P is *fully centralized* if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^{\mathcal{F}}$;
- P is *fully normalized* if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$; and
- P is *fully automized* if $\text{Aut}_S(P) \in Syl_p(\text{Aut}_{\mathcal{F}}(P))$.

Let $P \leq S$. Notice that the orbit $P^{\mathcal{F}}$ contains at least one fully centralized (respectively, normalized) element: since S is a finite group, take $Q \in P^{\mathcal{F}}$ such that $|C_S(Q)|$ (respectively, $|N_S(Q)|$) is maximal, so that Q is fully centralized (respectively, normalized). Let $\mathcal{F}^f$ be the set of all fully normalized subgroups of S. In the standard example $\mathcal{F}_S(G)$ it happens that $P \leq S$ is fully centralized (respectively, normalized) in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in Syl_p(C_G(P))$ (or $N_S(P) \in Syl_p(N_G(P))$, respectively). An element $u \in S$ is called *fully centralized* if $\langle u \rangle$ is a fully centralized subgroup.

2.2 Saturated Fusion Systems

Let $\mathcal{F}$ be a fusion system $\mathcal{F}$ on a p-group S. Given $P, Q \leq S$ and an isomorphism $\phi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ we define

$$N_\phi = \{g \in N_S(P) \mid c_g^\phi \in \mathrm{Aut}_S(Q)\}.$$

We say that Q is *receptive* if for all $P \in Q^{\mathcal{F}}$ and all $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, there exists $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ such that $\overline{\phi}|_P = \phi$. We point out that $C_S(P)P \leq N_\phi$. Also, N_ϕ is the largest subgroup of $N_S(P)$ to which $\phi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ may be extended. Indeed, suppose ϕ is extended to some subgroup P^* of $N_S(P)$. Then if $g \in P^*$ it follows that $c_g^\phi = c_{g^\phi} \in \mathrm{Aut}_S(Q)$.

We now able define saturated fusion systems.

Definition 2.3. A fusion system $\mathcal{F}$ over a p-group S is said to be *saturated* if the following two conditions hold:

(i) (Sylow Axiom) If $P \leq S$ is fully normalized in $\mathcal{F}$, then P is fully centralized and fully automized in $\mathcal{F}$.

(ii) (Extension Axiom) If Q is fully centralized in $\mathcal{F}$ then Q is also receptive in $\mathcal{F}$.

We observe that if $\mathcal{F}$ is a saturated fusion system on S then S is fully normalized, fully centralized, and $\mathrm{Aut}_S(S) = \mathrm{Inn}(S) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(S))$. The definition above is found in [BLO03] but there are several equivalent definitions in the literature due to Puig [Pui06], Stancu [Sta06], and Roberts and Shpectorov [RS09].

Theorem 2.4. *Let G be a finite group with $S \in Syl_p(G)$ for some prime p. Then $\mathcal{F}_S(G)$ is a saturated fusion system.*

Proof. This is principally an exercise in Sylow's Theorem and can be found in [Cra11b, Theorem 4.12] or [AKO11, Theorem I.2.3]. □

Another useful consequence of the saturation axioms is the following.

Lemma 2.5. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Assume $P \leq S$ and let $Q \in P^{\mathcal{F}}$ be fully normalized. Then there is a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(Q))$ such that $P^{\phi} = Q$.*

Proof. See [AKO11, Lemma I.2.6]. □

Given a saturated fusion system $\mathcal{F}$ on S we say that $\mathcal{F}$ is *realizable* if there exists a finite group G and a Sylow subgroup S of G such that $\mathcal{F} = \mathcal{F}_S(G)$; otherwise, $\mathcal{F}$ is called *exotic*. It turns out that when p is odd there are several known families of exotic systems. Some examples when p is odd may be found in the work of Ruiz and Viruel [RV04] and Clelland and Parker [CP10]. When $p = 2$, the case is a bit different. Solomon [Sol74] studied the Sylow 2-subgroups of $\mathrm{Spin}_7(q)$ for q odd. He showed there is no group G with a Sylow 2-subgroup S of $\mathrm{Spin}_7(q)$-type such that G has a single conjugacy class of involutions with $\mathcal{F}_S(\mathrm{Spin}_7(3)) \leq \mathcal{F}_S(G)$. However, Benson [Ben94] constructed 2-completed classifying spaces for the "non-existent simple groups" (i.e. fusion patterns) Solomon studied. Then Levi and Oliver [LO02] constructed a saturated 2-fusion system on the Sylow 2-subgroup of $\mathrm{Spin}_7(q)$ with the properties of Solomon's non-existent simple groups. It turns out these exotic systems are actually simple and, at present, are the only known examples of such systems for $p = 2$ (the definition of simple fusion systems is given in Section 2.4).

Let $\mathcal{F}$ be a fusion system on a p-group S and suppose $\mathcal{E}$ is a fusion system on $T \leq S$. Then we say $\mathcal{E}$ is a *subsystem* of $\mathcal{F}$ if $\mathcal{E}$ is a subcategory of $\mathcal{F}$, that is

$$\mathrm{Hom}_{\mathcal{E}}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \quad \text{for all } P, Q \leq T.$$

The intersection of two fusion systems on S is again a fusion system on S. With that in mind, let $\mathfrak{X}$ be a set of subsystems of $\mathcal{F}$ or a set of injective group homomorphisms between subgroups of S. Then the *fusion system generated by* $\mathfrak{X}$ is the smallest fusion system on S that contains $\mathfrak{X}$. This system, denoted $\langle\,\mathfrak{X}\,\rangle_S$, is the intersection of all systems containing $\mathfrak{X}$ and has as morphisms all composites of restrictions of homomorphisms in $\mathfrak{X} \cap \mathrm{Inn}(S)$ and their inverses.

We now prove a form of Alperin's Fusion Theorem for saturated fusion systems. While the fusion pattern in an abstract fusion system may be unwieldy, this theorem will tell us that *all* fusion is effected by a series of compositions among automorphisms of certain "local" subgroups. Given a fusion system $\mathcal{F}$ on a p-group S, we say that a subgroup $P \leq S$ is $\mathcal{F}$-*centric* if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$, and say P is $\mathcal{F}$-*radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. Let $\mathcal{F}^c$ and $\mathcal{F}^r$ be the set of all centric and radical subgroups in S, respectively; let $\mathcal{F}^{fcr}$ be the set of all subgroups in S that are fully normalized, $\mathcal{F}$-centric, and $\mathcal{F}$-radical.

Theorem 2.6. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S and let $P \leq S$. Then for all $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, there exist subgroups*

$$P = P_0,\ P_1,\ \ldots,\ P_m = P^{\phi} \quad \textit{and} \quad V_1,\ V_2,\ \ldots,\ V_m,$$

and $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(V_i)$*, such that*

(i) $P_{i-1}, P_i \leq V_i$*, and* $P_{i-1}^{\phi_i} = P_i, \quad 1 \leq i \leq m,$

(ii) $V_i \in \mathcal{F}^{fcr}$ *for all* i*, and*

(iii) $\phi = (\phi_1|_{P_0})(\phi_2|_{P_1}) \ldots (\phi_m|_{P_{m-1}})$.

In other words, $\mathcal{F} = \langle\, \mathrm{Aut}_{\mathcal{F}}(V) \mid V \in \mathcal{F}^{fcr} \,\rangle_S$.

Proof. We proceed by induction on the index of P in S. Clearly S is fully normalized and centric. By the first axiom of saturation, S is fully automized and $\mathrm{Out}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)/\,\mathrm{Inn}(S)$ is of order prime to p, that is, $O_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$. Therefore the theorem holds when $P = S$.

We now assume that $P < S$ and let P^ψ be a fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate of P with $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. If the theorem holds for ψ and $\phi^{-1}\psi \in \mathrm{Hom}_{\mathcal{F}}(P^\phi, P^\psi)$ then the theorem holds for $\phi = \psi(\phi^{-1}\psi)^{-1}$. Therefore we may assume that P^ϕ is fully normalized. Since P^ϕ is fully normalized, by Lemma 2.5 there exists a map $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(P^\phi))$ such that $P^\psi = P^\phi$. Since $N_S(P) > P$ the theorem holds for ψ on $N_S(P)$ by induction. It follows then that the theorem holds for ϕ if and only if it holds for $\psi^{-1}\phi \in \mathrm{Aut}_{\mathcal{F}}(P^\phi)$. We therefore reduce to the case that $P = P^\phi$ is fully normalized with $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$.

Since P is fully normalized, the theorem holds if P is $\mathcal{F}$-centric and $\mathcal{F}$-radical. So assume first that P is not $\mathcal{F}$-centric. Since $\mathcal{F}$ is saturated the Sylow Axiom gives us that P is fully centralized, so the Extension Axiom tells us that ϕ can be extended to $C_S(P)P \leq N_\phi$. Since P is not $\mathcal{F}$-centric $C_S(P) > Z(P)$ so that $C_S(P)P > P$ and the result holds by induction.

Finally, we assume that P is not $\mathcal{F}$-radical. Since P is fully normalized, the Sylow Axiom gives us that $\mathrm{Aut}_S(P) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(P))$ which combines with the fact that

$K = O_p(\mathrm{Aut}_{\mathcal{F}}(P)) > 1$ to yield

$$\mathrm{Inn}(P) < K \leq \mathrm{Aut}_S(P).$$

We therefore denote the preimage of K in S by $N_S^K(P)$ and observe that $P < N_S^K(P)$ as $\mathrm{Inn}(P) < K$. If $g \in N_S^K(P)$ then $\phi^{-1} c_g \phi \in K \leq \mathrm{Aut}_S(P)$ as K is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$. This implies that $N_S^K(P) \leq N_\phi$. Since $\mathcal{F}$ is a saturated fusion system we have that P is receptive and that ϕ extends to $N_S^K(P)$. Since $P < N_S^K(P)$ the result holds by induction. □

The above proof of Alperin's Fusion Theorem is from [BLO03] by way of [Cle06], though Puig proved a version in [Pui06].

Example 2.7. We present an immediate application of Alperin's Fusion Theorem: we show that there exist exactly three nonisomorphic saturated fusion systems on the dihedral group D_{2^n} where $n \geq 3$ (though we have not yet discussed morphisms between fusion systems). Let $S \cong D_{2^n}$ with $n \geq 3$. Then we shall show that the three saturated fusion systems (up to isomorphism) on S are

- $\mathcal{F}_S(S)$, the fusion system of D_{2^n}, or
- $\mathcal{F}_S(PGL_2(q))$, the fusion system of $PGL_2(q)$ for suitable odd q, or
- $\mathcal{F}_S(L_2(q))$, the fusion of $L_2(q)$ for suitable odd q.

Assume $\mathcal{F}$ is a saturated fusion system on

$$S = \langle\, r, s \mid r^{2^{n-1}} = s^2 = 1 \text{ and } r^s = r^{-1} \,\rangle.$$

Let $z = r^{2^{n-2}}$ be the central involution of S. We need to determine $\mathrm{Aut}_{\mathcal{F}}(P)$ for $P \leq S$ such that P *could* lie in $\mathcal{F}^{frc}$. All subgroups of S are cyclic or dihedral of order 2^m where $m \leq n$. The cyclic subgroups all have automorphism groups of 2-power order as do all the dihedral subgroups for $m > 2$ by Lemma 3.2. So for any $P \in \mathcal{F}^{frc}$ of these types, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_S(P)$. It remains only to consider the four-subgroups of S of which there are two conjugacy classes in S represented by $V_1 = \langle z, s \rangle$ and $V_2 = \langle z, rs \rangle$. We only need to determine the possibilities for $\mathrm{Aut}_{\mathcal{F}}(V_1)$ and $\mathrm{Aut}_{\mathcal{F}}(V_2)$: for if P is a four-group in S such that, say, $P^\phi = V_1$ then $\mathrm{Aut}_{\mathcal{F}}(P)^\phi = \mathrm{Aut}_{\mathcal{F}}(P^\phi) = \mathrm{Aut}_{\mathcal{F}}(V_1)$, so that $\mathrm{Aut}_{\mathcal{F}}(P) \cong \mathrm{Aut}_{\mathcal{F}}(V_1)$. We know that $\mathrm{Aut}(V_i) \cong S_3$ for $i = 1, 2$ so that $\mathrm{Out}_{\mathcal{F}}(V_i) \cong Z_2$ or S_3. When $\mathrm{Out}_{\mathcal{F}}(V_i) \cong Z_2$ for both $i = 1, 2$ then we get three classes of involutions and the system $\mathcal{F} \cong \mathcal{F}_S(S)$. If $\mathrm{Out}_{\mathcal{F}}(V_1) \cong S_3$ and $\mathrm{Out}_{\mathcal{F}}(V_2) \cong Z_2$ then we get two classes of involutions and $\mathcal{F} \cong \mathcal{F}_S(PGL_2(q))$ for suitable odd q; if we interchange 1 and 2 in the preceding statement we again get a fusion system $\mathcal{F} \cong \mathcal{F}_S(PGL_2(q))$. Finally, if $\mathrm{Out}_{\mathcal{F}}(V_i) \cong S_3$ for both $i = 1, 2$ then we have a single class of involutions and $\mathcal{F} \cong \mathcal{F}_S(L_2(q))$ for suitable odd q. For future reference, we remark that in all three systems, there are 2 classes of four-groups.

The example above shows, in particular, that there are no exotic fusion systems on a dihedral 2-group. A similar analysis (see [AKO11, Example I.3.8]) yields exactly 3 distinct saturated fusion systems on the quaternion groups Q_{2^n}, and exactly 4 distinct saturated fusion systems on the quasidihedral QD_{2^n} when $n > 3$ (we discuss these groups in the following chapter). In these cases, again, all the fusion systems are realizable.

2.3 Quotients and Normal Subsystems

We first introduce the normalizer and centralizer subsystems of a subgroup $Q \leq S$.

Definition 2.8. Let $\mathcal{F}$ be a fusion system on a p-group S and let $Q \leq S$.

- The *normalizer* $N_{\mathcal{F}}(Q)$ is the fusion system on $N_S(Q)$ with morphisms $\phi \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P, R)$ if and only if ϕ has an extension $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ with $Q^{\overline{\phi}} = Q$.

- The *centralizer* $C_{\mathcal{F}}(Q)$ is the fusion system on $C_S(Q)$ with morphisms $\phi \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, R)$ if and only if ϕ has an extension $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ with $\overline{\phi}|_Q = \mathrm{id}_Q$.

A subgroup Q is *normal* in $\mathcal{F}$ if $N_{\mathcal{F}}(Q) = \mathcal{F}$ and *central* if $C_{\mathcal{F}}(Q) = \mathcal{F}$. If Q_1 and Q_2 are normal (central) in $\mathcal{F}$ then it is easy to see that Q_1Q_2 is normal (central) in $\mathcal{F}$ as well. It therefore makes sense to define $O_p(\mathcal{F})$ and $Z(\mathcal{F})$ to be largest normal and central subgroups of $\mathcal{F}$. When working in a saturated fusion system, we would like the normalizer and centralizer subsystems to be saturated as well. While this is not always the case, Puig [Pui06, Proposition 2.15] proved this does hold in an important special case.

Proposition 2.9. *Suppose that $\mathcal{F}$ is a saturated fusion system on the p-group S and $Q \leq S$. Then $N_{\mathcal{F}}(Q)$ is saturated whenever Q is fully normalized and $C_{\mathcal{F}}(Q)$ is saturated whenever Q is fully centralized.*

We now define the concepts of strong and weak closure; these generalize to any fusion system $\mathcal{F}$ the corresponding group definitions in $\mathcal{F}_S(G)$.

Definition 2.10. Let $\mathcal{F}$ be a fusion system on a p-group S and let $Q \leq S$.

- Q is *weakly closed* in $\mathcal{F}$ if $Q^{\mathcal{F}} = \{Q\}$.

- Q is *strongly closed* in $\mathcal{F}$ if no element of Q is $\mathcal{F}$-conjugate to any element of $S - Q$.

We now discuss morphisms and quotients of fusion systems.

Definition 2.11. Let $\mathcal{F}$ and $\mathcal{G}$ be fusions systems on p-subgroups S and T. A *morphism* from $\mathcal{F}$ to $\mathcal{G}$ is a family $\Phi = (\phi, \phi_{P,Q})$ such that

(i) $\phi : S \longrightarrow T$ is a group homomorphism, and

(ii) For all $P, Q \leq S$, $\phi_{P,Q} : \mathrm{Hom}_{\mathcal{F}}(P,Q) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P^\phi, Q^\phi)$ is a map such that

$$\psi \circ \phi = \phi \circ (\psi \phi_{P,Q}) \quad \text{ for each } \psi \in \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

The morphism Φ is *surjective* if (1) ϕ is surjective as a group homomorphism and (2) for all $P, Q \leq S$, the map $\phi_{P_0,Q_0} : \mathrm{Hom}_{\mathcal{F}}(P_0, Q_0) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P^\phi, Q^\phi)$ is surjective, where P_0 and Q_0 are the preimages under ϕ in S of P^ϕ and Q^ϕ respectively. The morphism Φ is *injective* if ϕ is injective and each $\phi_{P,Q}$ is injective. The morphism Φ is an *isomorphism* when ϕ is an isomorphism and each $\phi_{P,Q}$ is bijective. We define the *kernel* of Φ to be $\ker(\phi)$. The kernel of Φ is a strongly closed subgroup of S. When $\mathcal{F}$ is saturated and Φ is surjective, it follows that image $\Phi(\mathcal{F})$ is a saturated fusion system as well by [AKO11, Lemma II.5.4].

Consider $\mathcal{F}$ and $\mathcal{G}$ above and let $\phi : S \longrightarrow T$ be a group isomorphism. We say that ϕ *preserves fusion* if $\mathrm{Hom}_{\mathcal{F}}(P,Q)^\phi = \mathrm{Hom}_{\mathcal{G}}(P^\phi, Q^\phi)$. In light of (ii) above, we see that there is a one-to-one correspondence between fusion preserving isomorphisms from S to T and isomorphisms of the fusion systems $\mathcal{F}$ and $\mathcal{G}$. When dealing with isomorphisms of fusion systems we will usually just refer to the isomorphism between the corresponding groups.

With the notion of morphism of fusion systems in hand, we now consider quotient subsystems. Let $\mathcal{F}$ be a fusion system on the p-group S and let $T \leq S$ be strongly closed in $\mathcal{F}$. We define the *factor system* $\mathcal{F}^+ = \mathcal{F}/T$ to be the category on $S^+ = S/T$ where for $T \leq P, Q \leq S$ we have that $\mathrm{Hom}_{\mathcal{F}^+}(P^+, Q^+)$ is the set of homomorphisms induced by $\mathrm{Hom}_{\mathcal{F}}(P, Q)$. By Lemmas 5.4 and 5.5 of [AKO11] we have that $\mathcal{F}/T$ is a fusion system and when $\mathcal{F}$ is saturated, so is $\mathcal{F}/T$.

Unfortunately, the kernel of a morphism between fusion systems is a subgroup rather than, say, a "normal" subsystem (which we define shortly). Ideally, we would like to realize the homomorphic image of a fusion system $\mathcal{F}$ on S as the quotient of $\mathcal{F}$ by some normal subsystem. In fact, it turns out that strongly $\mathcal{F}$-closed subgroups of S are in bijective correspondence with homomorphic images of $\mathcal{F}$.

Theorem 2.12. *Let $\mathcal{F}$ be a fusion system on a p-group S. The map $T \mapsto \mathcal{F}/T$ is a bijection between the set of strongly closed subgroups of S and the set of isomorphism classes of homomorphic images of $\mathcal{F}$.*

Proof. This [AKO11, Theorem II.5.14]. □

So far we have referred to "normal" subsystems but have not actually defined them. This is due in part to the existence of several competing notions of what normal should mean. From weakest to strongest, we consider $\mathcal{F}$-invariance due to Puig [Pui06], weak normality due to Linckelmann [Lin06] and Oliver [Oli10], and what we call normality due to Aschbacher [Asc08].

Definition 2.13. Let $\mathcal{F}$ be a saturated fusion system on a p-group S and let $\mathcal{E}$ be a fusion subsystem of $\mathcal{F}$ on a strongly $\mathcal{F}$-closed subgroup T.

- $\mathcal{E}$ is *$\mathcal{F}$-invariant* if for each $P \leq Q \leq T$, $\psi \in \mathrm{Hom}_{\mathcal{E}}(P, Q)$, and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, we have $\psi^{\phi} \in \mathrm{Hom}_{\mathcal{E}}(P^{\phi}, Q^{\phi})$.

- $\mathcal{E}$ is *weakly normal* if $\mathcal{E}$ is saturated and $\mathcal{F}$-invariant.

- $\mathcal{E}$ is *normal* if $\mathcal{E}$ is weakly normal and if each $\phi \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to $\overline{\phi} \in$

$\mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[C_S(T), \overline{\phi}] \leq Z(T)$ where

$$[C_S(T), \overline{\phi}] = \langle\, g^{-1}g^{\overline{\phi}} \mid g \in C_S(T)\,\rangle.$$

The concept of $\mathcal{F}$-invariance is appealing in that the definition is intuitive and analogous to that of a normal subgroup. That said, an $\mathcal{F}$-invariant subsystem need not be saturated. For example, suppose $\mathcal{E}$ is the full subcategory of $\mathcal{F}$ on some strongly closed subgroup $T \leq S$. By default $\mathcal{E}$ is $\mathcal{F}$-invariant but need not be saturated. By definition $\mathrm{Aut}_{\mathcal{E}}(T) = \mathrm{Aut}_{\mathcal{F}}(T)$ and since $\mathcal{F}$ is saturated and T is fully normalized, $\mathrm{Aut}_S(T) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(T))$. Then if $\mathrm{Inn}(T) < \mathrm{Aut}_S(T)$ it follows that $\mathrm{Inn}(T) = \mathrm{Aut}_T(T)$ is *not* Sylow in $\mathrm{Aut}_{\mathcal{E}}(T) = \mathrm{Aut}_{\mathcal{F}}(T)$. Since T is fully $\mathcal{E}$-normalized as well, it follows that $\mathcal{E}$ violates the Sylow Axiom for saturation.

Examples I.6.3 and II.7.3 of [AKO11] give instances in which a saturated fusion system $\mathcal{F}$ contains a weakly normal subsystem $\mathcal{E}$ which is not normal in $\mathcal{F}$. At this point it is natural to wonder what is "wrong" with weak normality; after all, this subsystem is closed under conjugation and is saturated. One issue, as Aschbacher explains in [AKO11, II.7], is that the extra condition for normality is necessary for the theory of saturated fusion systems to provide a satisfactory extension of the local theory of finite groups. In particular, this condition is required to ensure that if $\mathcal{F}$ is in the important class of *constrained* fusion systems, and $\mathcal{F}$ is realizable by the finite group G, and $\mathcal{E}$ is normal in $\mathcal{F}$, then $\mathcal{E}$ is realizable by some $H \trianglelefteq G$.

2.4 Simple Fusion Systems

A fusion system is said to be *simple* if it possesses no nontrivial normal subsystems. It is the potential classification of simple (saturated) fusion systems with which this thesis concerns itself. First, simple groups do not necessarily give rise to simple fusion systems. The fusion system on A_5 has a normal subsystem for each prime $p = 2, 3, 5$. In fact, any fusion system on a group containing a strongly closed abelian p-group contains a normal subsystem at p. The latter occurs, in particular, when a Sylow p-subgroup of a group G is abelian.

In [AKO11, II.14] Aschbacher lays out potential paths by which the simple saturated 2-fusion systems might be classified, which we summarize here. The motivation is, aside from independent interest, that the Classification of Finite Simple Groups might in some ways be simplified by first proving theorems on simple fusion systems. The program he suggests is very much analogous to that followed in the Classification. Before pursuing this further, we need several definitions.

In the study of fusion in groups $O^p(G)$, the smallest normal subgroup of G of p-power index, plays a significant role. There exists a natural fusion systems analog to this subgroup, $O^p(\mathcal{F})$, which we postpone defining precisely until Chapter 4 and our discussion of transfer. In the meantime, a fusion system is *quasisimple* if $O^p(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}/Z(\mathcal{F})$ is simple. The *components* of $\mathcal{F}$ are the subnormal, quasisimple subsystems of $\mathcal{F}$. The fusion system $\mathcal{F}$ is *constrained* if $O_p(\mathcal{F})$ is centric. We say that $\mathcal{F}$ is of *component type* if there exists a fully normalized subgroup X of order p such that $C_{\mathcal{F}}(X)$ has a component. On the other hand, we say that $\mathcal{F}$ is of *characteristic p-type* if $N_{\mathcal{F}}(U)$ is constrained for each fully normalized subgroup U. As one would hope, there is a dichotomy theorem for fusion systems similar to that for groups.

Theorem 2.14. *Let $\mathcal{F}$ be a saturated fusion system on a p-group. Then either $\mathcal{F}$ is of either characteristic p-type or $\mathcal{F}$ is of component type.*

Proof. This is [AKO11, Theorem II.14.3]. □

A natural approach to classifying the simple saturated 2-fusion systems would be to classify all simple 2-fusion systems of component type and characteristic 2-type.

This thesis is concerned with fusion systems of component type so we discuss their possible classification as laid out in [AKO11]. The first step would be to prove an analog of Aschbacher's Classical Involution Theorem which would characterize most 2-fusion systems of Lie type in odd characteristic. Aschbacher has already begun work on this problem by considering what he defines to be *quaternion fusion packets.* Next, one would prove a theorem analogous to Aschbacher's Component Theorem: that is (with a few exceptions), a 2-fusion system of component type contains a "standard component". The final step would be to the solve the various standard form problems, that is, classify fusion systems with a standard component of various isomorphism types.

It is this final step to which we contribute in this thesis. At present, there is no accepted notion of a standard component in fusion systems; however, Aschbacher has given a definition of a tightly embedded subsystem and begun the classification of such subsystems. We would like a "standard" component then to have a tightly embedded centralizer in $\mathcal{F}$. Unfortunately, there is no accepted notion of normalizer or centralizer of *subsystems* to date. In the special case when a subsystem $\mathcal{E}$ is normal in $\mathcal{F}$ Aschbacher defines the centralizer of $\mathcal{E}$ in $\mathcal{F}$ denoted by $C_{\mathcal{F}}(\mathcal{E})$. The definition of this subsystem is quite technical but proves to have the basic properties we would like.

Theorem 2.15. *If $\mathcal{E}$ is a normal subsystem on T of the saturated subsystem $\mathcal{F}$ on S then the set of all subgroups $Y \leq C_S(T)$ such that $\mathcal{E} \leq C_{\mathcal{F}}(Y)$ has a largest member denoted by $C_S(\mathcal{E})$. Moreover, $C_{\mathcal{F}}(\mathcal{E})$ is a normal subsystem on $C_S(\mathcal{E})$.*

Proof. This is [Asc11a, 6.7]. □

Before outlining the problems considered in this thesis we point out there is still question as to what the right partition of simple saturated fusion systems should be.

As with the original Classification of Finite Simple Groups, certain technical difficulties arise when considering subsystems of component type, so it may be best to slightly alter the partition of simple fusion systems. In particular, the way forward might be to restrict to components with additional hypotheses – what Aschbacher calls *even component type* and *Baumann component type* – and prove a variant of Aschbacher's Standard Form Theorem for just these types of components. Fortunately, the eventual partition will not affect our work in any case as we put no additional (even/Baumann) restrictions on our hypotheses. We also point out that the program outlined above makes no mention of the B-conjecture or signalizer functors which present so much difficulty in the original Classification, because fusion systems do not account for cores (normal subgroup of p'-order). While this property of fusion systems makes our work much more manageable, in order to apply the theory to obtain a full classification of groups with a given fusion system, presumably these cores must eventually be accounted for.

2.5 The Main Results of the Thesis

In this thesis we consider two standard form problems.

2.5.1 The $SL_2(q)$ Standard Form Problem

First, we consider the situation in which a saturated fusion system has a "standard" component of type $SL_2(q)$, q odd. In particular we prove the following.

Theorem 1. Let $\mathcal{F}$ be a simple saturated fusion system on a 2-group S with R a generalized quaternion subgroup of S. Assume further that

(1) R is weakly closed in S with respect to $\mathcal{F}$, and

(2) R is strongly closed in $C_{\mathcal{F}}(y)$ for all involutions $y \in C_S(R)$.

Then either $\mathcal{F}$ is the fusion system of $L_3(q)$ or $G_2(q)$ for some odd q, or S contains a strongly $\mathcal{F}$-closed quasidihedral subgroup.

In fact, we will say even more about $\mathcal{F}$. First, we suppose $|R| = 2^n$. Then – employing the notation from Section 3.1 – in the case S is of $G_2(q)$-type it will follow that $S \cong (Q_1 * Q_2)\langle\, t\,\rangle$ where $Q_1 \cong Q_2 \cong Q_{2^n}$ and $R = Q_1$ or Q_2 so that $|S| = 2^{2n}$. When $\mathcal{F}$ is the fusion system of $L_3(q)$, $S \cong Z_{2^{n-1}} \wr Z_2$ is wreathed so that $|S| = 2^{2n-1}$. Finally, when S contains a strongly closed quasidihedral subgroup P we will show that $P = R\langle\, z^{\phi}\,\rangle$ for an appropriate $\phi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, z\,\rangle, S)$, where $\langle\, z\,\rangle = Z(R)$, so that $|P| = 2^{n+1}$.

To get that the 2-groups described above give rise to the fusion systems of $L_3(q)$ and $G_2(q)$ we invoke a theorem of Oliver in [Oli11]. Even without this result, we could determine the fusion system $\mathcal{F}$ by invoking the work of Gorenstein and Harada [GH71] and the fusion pattern we will ultimately determine for these 2-groups.

The outline of this part of the thesis is as follows: In Chapter 5 we prove a group-theoretic version of Theorem 1, basically assuming $S \in Syl_2(G)$ for some (fusion) simple group G. This is then used as a template for the (independent) fusion systems proof given in Chapter 6. While the group-theoretic *result* has long been known (see [Foo76a],[Foo76b]), our proof in Chapter 5 is original and is strictly in terms of fusion arguments. It is important to note that our Theorem 1 *subsumes* the group theoretic standard form problem for $SL_2(q)$, even though our theorem makes no mention of a specific component or tight embedding – we demonstrated this in the previous chapter. This generality makes us confident that our hypothesis will be compatible with any future definition of "standard form" in the theory of fusion systems.

2.5.2 The $L_2(q)$ Standard Form Problem

The second problem treats the case when our standard component is of type $L_2(q)$, q odd. We note that if $q \equiv \pm 3(\mod)8$ then by order considerations a Sylow 2-subgroup of $L_2(q)$ is a four-group, and by Lemma 4.2 such a fusion system is not quasi-simple and so cannot be a component.

The situation for components of type $L_2(q)$ is more complicated than for $SL_2(q)$ and we first provide some additional background before stating our second theorem. Because for odd q there can be an involution in $\mathrm{Aut}(L_2(q))$ that centralizes a Sylow 2-subgroup of $L_2(q)$ and because the 2-rank of $L_2(q)$ is 2, tight embedding does not imply the same nice (weak closure) reduction as in our first problem. In this case, we will need part of Aschbacher's definition of tightly embedded fusion systems from [Asc11b].

Definition 2.16. Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Define $\mathfrak{A}(P)$ to be the set of maps $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that P^α is fully normalized. Also, if $\mathcal{Q}$ is a subsystem of $\mathcal{F}$ on $Q \leq S$ and $X \leq S$ then define

$$X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} = \{X^{\varphi\phi} \mid \varphi \in \mathrm{Aut}_{\mathcal{F}}(Q) \quad \text{and} \quad \phi \in \mathrm{Hom}_{\mathcal{Q}}(X^\varphi, Q)\}.$$

A saturated subsystem $\mathcal{Q}$ on a fully normalized subgroup $Q \leq S$ is *tightly embedded* in $\mathcal{F}$ if it satisfies the following three conditions.

(T1) For each $1 \neq P \in Q^f$ and each $\alpha \in \mathfrak{A}(P)$, we have $O^{p'}(N_{\mathcal{Q}}(P))^\alpha \trianglelefteq N_{\mathcal{F}}(P^\alpha)$.

(T2) For each subgroup X of order p in Q , $X^{\mathcal{F}} \cap Q = X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}}$.

(T3) $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(\mathcal{Q})$.

So if we consider a saturated fusion system $\mathcal{F}$ with fully centralized involution t

such that $C_{\mathcal{F}}(t)$ has a "standard" component $\mathcal{K}$, which subsystem of $\mathcal{F}$ should we hypothesize to be tightly embedded? By analogy with groups, we would like the answer to be "$C_{\mathcal{F}}(\mathcal{K})$", but that object is not even defined for a general component $\mathcal{K}$. However in the group case a standard component is normal in an involution centralizer, so if we hypothesize that $\mathcal{K}$ is *normal* in $\mathcal{C} = C_{\mathcal{F}}(t)$ then we *can* define $C_{\mathcal{C}}(\mathcal{K})$ by Theorem 2.15. So in our analysis we will first impose the normality condition of $\mathcal{K}$ in $\mathcal{C}$ and then impose conditions on $C_{\mathcal{C}}(\mathcal{K})$ that are weaker than (T1)-(T3), so $C_{\mathcal{C}}(\mathcal{K})$ is "almost tightly embedded" as part of our working definition of standard.

The other issue we would like to resolve before stating our second theorem pertains to $L_2(q)$ specifically. In particular, when $q > 9$ we will see in Lemma 3.19 that if U is a four-group acting on $L = L_2(q)$ then $\Gamma_{U,1}(L) = L$. When L is standard in a group G this property together with the standard form properties implies that no four-group centralizing L may conjugate out of the centralizer of L and still normalize L (see Lemma 7.3). We refer to this implication as the Γ_1-property. In the fusion systems situation we would like the centralizer of our component to have this property as well, but at this point it is unclear how to translate the group-theoretic notion of $\Gamma_{U,1}(L)$ into fusion systems.

In light of these considerations, we combine some of the properties of Aschbacher's definition of tightly embedded with our version of the Γ_1-property for fusion systems into one definition.

Definition 2.17. Let $\mathcal{F}$ be a saturated fusion system on a 2-group S. A saturated subsystem $\mathcal{Q}$ on a subgroup Q of S is Γ_1-*embedded* in $\mathcal{F}$ if it satisfies the following conditions.

(S1) For any fully $\mathcal{Q}$-centralized involution $w \in Q$, we have that $C_Q(w)$ is strongly closed in $C_{\mathcal{F}}(w)$.

(S2) For any involution $w \in Q$ and $\mathcal{F}$-conjugate $w^\phi \in Q$, there exists a map $\alpha \in \mathrm{Hom}_{\mathcal{Q}}(\langle w \rangle, Q)$ such that $w^\phi = w^\alpha$.

(S3) If $W \leq Q$ is a four-group such that $W^\phi \leq N_S(Q)$ for some $\phi \in \mathrm{Hom}_{\mathcal{F}}(W, N_S(Q))$ then $W^\phi \cap Q \neq 1$.

Condition (S3) is (a weaker form of) the aforementioned Γ_1-property. Observe that (S1) is a weaker version of Aschbacher's (T1); (T1) guarantees that $C_Q(w)^\alpha$ is strongly closed in $C_{\mathcal{F}}(w^\alpha)$ for $\alpha \in \mathfrak{A}(\langle w \rangle)$, and this implies that $C_Q(w)$ is strongly closed in $C_{\mathcal{F}}(w)$. Condition (S2) is just (T2), and we omit Aschbacher's (T3) entirely. We are finally able to state our hypothesis and hence our working fusion systems definition of a standard component of $L_2(q)$ type.

Hypothesis 2. Let $\mathcal{F}$ be a simple saturated fusion system on a 2-group S. Suppose that $\mathcal{F}$ contains a nontrivial subgroup Q and a subsystem $\mathcal{K}$ on $R \leq S$ of type $L_2(q)$, with q odd, such that $|R| = 2^m$ and $m \geq 3$. Suppose further that

(1) $\mathcal{K}$ is a normal component of $C_{\mathcal{F}}(z)$ for every fully $\mathcal{F}$-centralized involution $z \in Z(Q)$,

(2) $C_{\mathcal{C}}(\mathcal{K})$ is a fusion system on Q, where z is any involution as in (1) with $\mathcal{C} = C_{\mathcal{F}}(z)$, and

(3) $C_{\mathcal{C}}(\mathcal{K})$ is Γ_1-embedded for some z as in (1).

With the exception of the Γ_1-property, we believe our working definition of standard above should be compatible with any future definition of standard. Notice that condition (2) forces Q to be the full 2-part of the centralizer of $\mathcal{K}$. We now state our second theorem.

Theorem 2. Suppose that Hypothesis 2 holds. If Q has rank at least 2 then one of the following holds:

(i) Q is $\mathcal{F}$-strongly closed, or

(ii) Q is a dihedral subgroup of index 2 contained in a strongly closed dihedral group.

As with Theorem 1, we first prove a “fusion friendly”, group-theoretic version of the theorem in Chapter 7, and subsequently we prove Theorem 2 in Chapter 8. As before, the group-theoretic standard form theorem was already established [Foo78] but again, our new proof relies only on fusion arguments. A case of the $L_2(q)$ standard form problem in which Q is of rank 1 is handled by Justin Lynd in his thesis [Lyn12], to which we consider our work to be complementary.

Chapter 3

Preliminary Group Theory Lemmas

In this chapter we establish a number of group theory lemmas that figure prominently in the proofs of our main theorems.

3.1 The Automorphism Groups of D_{2^n} and Q_{2^n}

In this thesis we will require various results on the structure of the automorphism groups of both the dihedral groups and the generalized quaternion groups. Let

$$D_{2^n} = \langle s, r \mid r^{2^{n-1}} = s^2 = 1, r^s = r^{-1} \rangle \text{ for } n \geq 2,$$

and

$$Q_{2^n} = \langle s, r \mid r^{2^{n-1}} = s^4 = 1, r^{2^{n-2}} = s^2, r^s = r^{-1} \rangle \text{ for } n \geq 3,$$

be the *dihedral* and *generalized quaternion* groups, respectively, of order 2^n. Our main aim of this section is to determine $\text{Aut}(Q_{2^n})$ and $\text{Aut}(D_{2^n})$ but only after we have defined several other 2-groups that will be of importance throughout:

- The *quasidihedral* group QD_{2^n} of order 2^n

$$QD_{2^n} = \langle\, s, r \mid r^{2^{n-1}} = s^2 = 1, r^s = r^{2^{n-2}-1} \,\rangle \text{ for } n \geq 4.$$

- The *wreath product* $Z_{2^n} \wr Z_2$ of order 2^{2n+1}, given by

$$Z_{2^n} \wr Z_2 = (Z_{2^n} \times Z_{2^n}) \rtimes Z_2 = (\langle\, a \,\rangle \times \langle\, b \,\rangle)\langle\, t \,\rangle$$

  where $\langle\, a \,\rangle \cong \langle\, b \,\rangle \cong Z_{2^n}$. Here t is a wreathing involution with action given by $a^t = b$ and $b^t = a$.

- Let $Q_1 \cong Q_2$ be generalized quaternion groups of order 2^n. Let $Q_1 * Q_2$ be their central product with common center and let t be an involution normalizing both Q_i such that

$$Q_i\langle\, t \,\rangle \cong QD_{2^{n+1}}, \text{ for } i = 1, 2.$$

  Then we shall say $(Q_1 * Q_2)\langle\, t \,\rangle$ is of $G_2(q)$-*type* as this group is isomorphic to a Sylow 2-subgroup of $G_2(q)$, for suitable q odd [GH71].

We begin our determination of $\mathrm{Aut}(D_{2^n})$ with the observation that D_4 is a 2-dimensional vector space over the field of two elements. Thus $\mathrm{Aut}(D_4) \cong GL_2(2) \cong S_3$ which we record as our first lemma.

Lemma 3.1. *If* $n = 2$ *then* $\mathrm{Aut}(D_{2^n}) \cong S_3$.

Lemma 3.2. *If* $n \geq 3$ *then* $|\,\mathrm{Aut}(D_{2^n}))| = 2^{2n-3}$ *and* $\mathrm{Aut}(D_{2^n}) \cong \mathrm{Hol}(Z_{2^{n-1}})$.

Proof. Let $D = D_{2^n}$. First, any automorphism of D must map r to some other element of order 2^{n-1} and all such elements are in $\langle r\rangle$, that is, $\langle r\rangle \,\mathrm{char}\, D$. Thus there are 2^{n-2} possible destinations for r. Now s must be mapped to an element of order 2 that inverts $\langle r\rangle$, and the set of such elements is $D - \langle r\rangle$ so there are 2^{n-1} possible destinations for s. As these choices satisfy all the presentation relations, we conclude that $|\,\mathrm{Aut}(D)| = 2^{n-2}2^{n-1}$ as claimed.

Now consider the subgroup H of $\mathrm{Aut}(D)$ that fixes s. Then H acts faithfully as the group of automorphisms of $\langle r \rangle$ so $H \cong \mathrm{Aut}(Z_{2^{n-1}}) \cong Z_2 \times Z_{2^{n-3}}$ [DF04, Exercises 22-23, Section 2.3]. Now let $\phi \in \mathrm{Aut}(D)$ be defined as follows:

$$\phi(s) = rs \text{ and } \phi(r) = r.$$

So it follows that $\langle \phi \rangle$ is the subgroup of $\mathrm{Aut}(D)$ fixing r and $|\langle \phi \rangle| = 2^{n-1}$. Since $\langle r \rangle$ is characteristic in D, $\langle \phi \rangle = C_{\mathrm{Aut}(D)}(\langle r \rangle)$ is normal in $\mathrm{Aut}(D)$, so by order considerations $\mathrm{Aut}(D) = \langle \phi \rangle H$ as $\langle \phi \rangle \cap H = 1$. Finally, we show that H acts on $\langle \phi \rangle$. Let $\psi \in H$ so then $\psi(r) = r^\alpha$ for some odd integer α with inverse α^{-1} in $(\mathbb{Z}/2^{n-1}\mathbb{Z})^\times$. Then

$$\phi^{\psi^{-1}}(s) = \psi(\phi(\psi^{-1}(s))) = \psi(\phi(s)) = \psi(rs) = r^\alpha s = \phi^\alpha(s) \text{ and}$$

$$\phi^{\psi^{-1}}(r) = \psi(\phi(\psi^{-1}(r))) = \psi(\phi(r^{\alpha^{-1}})) = \psi(r^{\alpha^{-1}}) = r = \phi^\alpha(r)$$

so we have that H acts faithfully on $\langle \phi \rangle$. Note that we have also shown that $C_{\mathrm{Aut}(D)}(\langle \phi \rangle) = \langle \phi \rangle$. We therefore conclude that $\mathrm{Aut}(D) \cong Z_{2^{n-1}} \rtimes (Z_2 \times Z_{2^{n-3}}) \cong \mathrm{Hol}(Z_{2^{n-1}})$ as claimed. □

Next, we determine $\mathrm{Aut}(Q_{2^n})$.

Lemma 3.3. *If* $n = 3$ *then* $\mathrm{Aut}(Q_8) \cong S_4$.

Proof. This is Exercise 6.3.9 in [DF04]. □

It turns out that $\mathrm{Aut}(D_{2^n}) \cong \mathrm{Aut}(Q_{2^n})$ for $n > 3$. The proof of this fact is identical to Lemma 3.2 with "element of order 2" replaced by "element of order 4".

Lemma 3.4. *If* $n > 3$ *then* $|\mathrm{Aut}(Q_{2^n})| = 2^{2n-3}$ *and* $\mathrm{Aut}(Q_{2^n}) \cong \mathrm{Hol}(Z_{2^{n-1}})$.

We record here, for frequent future use, some facts on the structure of $\mathrm{Aut}(Z_{2^n})$ when $n \geq 3$.

Lemma 3.5. *If $n \geq 3$, then $Aut(Z_{2^n}) \cong Z_2 \times Z_{2^{n-2}}$ contains exactly three involutions, denoted as inversion ($a \mapsto a^{-1}$), modular ($a \mapsto a^{1+2^{n-1}}$), and quasidihedral ($a \mapsto a^{-1+2^{n-1}}$) for $a \in Z_{2^n}$. Furthermore, if $n \geq 4$ then the modular involution is the only one rooted in* $\mathrm{Aut}(Z_{2^n})$.

Proof. This follows from Exercises 22 − 23 in Section 2.3 of [DF04]. □

The last result we need on $\mathrm{Aut}(D_{2^n}) \cong \mathrm{Aut}(Q_{2^n})$ is that $Z_2 \times Z_{2^{n-1}}$ cannot act faithfully on Q_{2^n} or D_{2^n}. For the proof of the following lemma we maintain the notation from Lemma 3.2.

Lemma 3.6. $\mathrm{Aut}(D_{2^n}) \cong \mathrm{Aut}(Q_{2^n})$ *has no subgroup isomorphic to $Z_2 \times Z_{2^{n-1}}$ for $n \geq 3$.*

Proof. If $n = 3$ then $\mathrm{Aut}(Q_{2^n}) \cong S_4$ which has Sylow 2-subgroups isomorphic to $D_8 \cong \mathrm{Aut}(D_8)$. Clearly D_8 contains no subgroup isomorphic to $Z_2 \times Z_4$. Now let $n > 3$ and suppose to the contrary that $\mathrm{Aut}(Q_{2^n})$ has some subgroup $K \cong Z_2 \times Z_{2^{n-1}}$ and let $K_0 = K \cap \langle \phi \rangle$. By order considerations we see that K_0 contains a cyclic subgroup of order at least 4. Since $\langle \phi \rangle$ is self-centralizing, we have that K/K_0 acts faithfully on $\langle \phi \rangle$ and we argue that it is cyclic. Otherwise K/K_0 would contain all three involutions acting on $\langle \phi \rangle$. Among these involutions is inversion which does not centralize the cyclic subgroup of order 4. This shows K/K_0 is cyclic, so for some $\sigma \in \mathrm{Aut}(Q_{2^n})$ we have that

$$\overline{K} \cong K/K_0 \cong \langle \overline{\sigma} \rangle.$$

Now $\langle \overline{\sigma} \rangle \leq \mathrm{Aut}(\langle \phi \rangle) = \langle \rho \rangle \times \langle \tau \rangle$ where ρ acts by inversion and $\phi^\tau = \phi^5$ [DF04, Exercises 2.3.22-23]. Since $\overline{\sigma}$ centralizes the subgroup of order 4 in $\langle\, \phi \,\rangle$ we have that $\overline{\sigma} \in \langle \tau \rangle$. So let $|\overline{\sigma}| = 2^{n-k}$ where $k = |K_0|$ and so we may choose the generator σ to satisfy

$$\overline{\sigma} = \tau^{2^{n-3-(n-k)}} = \tau^{2^{k-3}}.$$

It follows easily by induction that

$$|C_{\langle\phi\rangle}(\tau^{2^i})| = 2^{i+2}$$

and from this we get that

$$|C_{\langle\phi\rangle}(\overline{\sigma})| = 2^{k-3+2} = 2^{k-1}.$$

This contradicts the fact that

$$|C_{\langle\phi\rangle}(\overline{\sigma})| \geq |K_0| = 2^k$$

thereby completing the proof.

□

3.2 Notes on Some 2-groups

For convenience we begin by listing some properties of quasidihedral groups. For a group P, let $\Phi(P)$ denote the *Frattini* subgroup of P, that is, the intersection of all the maximal subgroups of P.

Lemma 3.7. *Let $P \cong QD_{2^n}$. Then the following statements hold:*

(1) $\Phi(P) = P'$ is cyclic of order 2^{n-2},

(2) $|Z(P)| = 2$, and

(3) the maximal subgroups of P are cyclic, generalized quaternion, or dihedral. Furthermore, P contains no proper quasidihedral subgroups.

(4) Any 2-group that has a cyclic subgroup of index 2 and an element outside that subgroup inducing a quasidihedral automorphism on it is a quasidihedral group.

Proof. This is [Gor80, 5.4.3]. □

We will also need the following property of $Z_{2^n} \wr Z_2$.

Lemma 3.8. *$Z_{2^n} \wr Z_2$ does not contain a normal subgroup isomorphic to $QD_{2^{n+1}}$ for $n > 3$.*

Proof. Set $P = (\langle a \rangle \times \langle b \rangle)\langle t \rangle$ where $\langle a \rangle \cong \langle b \rangle \cong Z_{2^n}$. Here t is a wreathing involution with action given by $a^t = b$ and $b^t = a$. Direct computation shows that P has three classes of involutions: $\{a^{2^{n-1}}b^{2^{n-1}}\}, \{a^{2^{n-1}}, b^{2^{n-1}}\}$, and $\{a^i b^{-i} t \mid 0 \leq i < 2^n\}$. These classes are of sizes $1, 2$, and 2^n, respectively. The group $QD_{2^{n+1}}$ has one central involution and 2^{n-1} non-central involutions. If P contains a normal subgroup H then H is the union of conjugacy classes in P. By the orders of the conjugacy classes of involutions in P, it cannot happen that $H \cong QD_{2^{n+1}}$. □

In the remainder of this section we shall develop some of the structure of the 2-groups of $G_2(q)$-type which will describe some of our "target" groups in the chapters that follow.

Let a and b be commuting elements of order 2^n with $n \geq 2$, and let

$$A = \langle a, b \rangle \cong Z_{2^n} \times Z_{2^n} \qquad \text{and} \qquad z = (ab)^{2^{n-1}}.$$

Let u, v be elements normalizing A and inducing the following automorphisms of order 2:

$$a^u = b, \quad b^u = a \qquad \text{and} \qquad a^v = b^{-1}, \quad b^v = a^{-1}.$$

Note that these automorphisms commute in their action on A, and uv inverts A. We may therefore choose u, v with the additional properties that

$$[u, v] = 1, \quad u^2 = v^2 = z, \qquad \text{and so } (uv)^2 = u^2v^2 = z^2 = 1. \tag{3.1}$$

Finally, let $P = \langle\, A, u, v \,\rangle$ so $|P| = 2^{2n+2}$. In this situation we shall see the following:

Lemma 3.9. *With the notation above we have that* $P = (R_1 * R_2)\langle\, t \,\rangle$ *where* $R_1 = \langle\, ab^{-1}, u \,\rangle$ *and* $R_2 = \langle\, ab, v \,\rangle$, $t = uva$, *and the following hold:*

(i) $R_i \cong Q_{2^{n+1}}$ *for* $i = 1, 2$,

(ii) t *is an involution normalizing both* Q_i, *and*

(iii) $R_i\langle\, t \,\rangle \cong QD_{2^{n+2}}$, *for* $i = 1, 2$.

In other words P *is of* $G_2(q)$*-type.*

Proof. First note that $\langle\, a, u \,\rangle = A\langle\, u \,\rangle \cong Z_{2^n} \wr Z_2$ where the involution $u_1 = a^{2^{n-1}}u$ interchanges $\langle\, a \,\rangle$ and $\langle\, b \,\rangle$. Likewise $\langle\, a, v \,\rangle = A\langle\, v \,\rangle \cong Z_{2^n} \wr Z_2$ with $v_1 = a^{2^{n-1}}v$ a wreathing involution. Each of these wreath products is of index 2 in P.

Next observe that each R_i is a generalized quaternion group of order 2^{n+1}. Furthermore, by construction

$$R_1R_2 = R_1 * R_2$$

which is a central product with common center $\langle\, z \,\rangle$. By order considerations the above subgroup is of index 2 in P.

Next consider $D_2 = \langle\, ua, v \,\rangle$: First compute that

$$(ua)^2 = uaua = u^2u^{-1}aua = zba = abz \tag{3.2}$$

and so $\langle\, ab \,\rangle$ is a subgroup of index 2 in $\langle\, ua \,\rangle$. Thus $\langle\, ua \,\rangle \cong Z_{2^{n+1}}$ and z is the unique involution in it. Also by (2):

$$(ua)^v = u(a^v) = ub^{-1} = (ua)(ua)^{-2}z = (ua)^{-1}z.$$

Thus D_2 has a cyclic subgroup of index 2 and an element of D_2 outside this cyclic subgroup acts as a quasidihedral automorphism on it; by Lemma 3.7 this proves

$$D_2 \cong QD_{2^{n+2}}.$$

Parallel calculations show that for $D_1 = \langle\, va, u \,\rangle$ we have $D_1 \cong QD_{2^{n+2}}$. Note that by (3.2) and the corresponding calculation for D_1 we have

$$D_i \text{ contains } R_i \text{ as a subgroup of index 2 for both } i = 1, 2.$$

Finally, let $t = uva$. Note that by (1), since uv is an involution that inverts a, t is an involution. Moreover, since P is generated by u, v and a one sees that $P = R_1 R_2 \langle\, t \,\rangle$. Finally,

$$R_1 \langle\, t \,\rangle = \langle\, ab^{-1}, u, uva \,\rangle \geq \langle\, va, u \,\rangle = D_1$$

so by order considerations we get $D_1 = R_1 \langle\, t \,\rangle \cong QD_{2^{n+2}}$. Likewise $R_2 \langle\, t \,\rangle = D_2 \cong QD_{2^{n+2}}$. Thus

$$P = (R_1 * R_2)\langle\, t \,\rangle \qquad \text{where} \qquad R_i \langle\, t \,\rangle \cong QD_{2^{n+2}}, \text{ for both } i = 1, 2,$$

as desired. □

3.3 Miscellaneous Lemmas

In this section we shall collect various lemmas for future reference.

Lemma 3.10. *(Suzuki's Lemma) Let P be a 2-group containing an involution t such that $C_P(t)$ is a four-group. Then P is a dihedral or quasidihedral group; and if P contains a quaternion subgroup, then P must be quasidihedral.*

Proof. This is Lemma 4 in [Suz51]. □

Lemma 3.11. *(Fitting's Lemma) Let P be a p-group and $A \leq \mathrm{Aut}(P)$ where A is of order prime to p. Then*

$$P = [P, A]C_P(A).$$

Moreover, if $[P, A]$ is abelian, then $[P, A] \cap C_P(A) = 1$ so $P = [P, A] \rtimes C_P(A)$.

Proof. That $P = [P, A]C_P(A)$ is the content of [Gor80, 5.3.5]. If $[P, A]$ is abelian then the abelian case of Fitting's Lemma [Gor80, 5.2.3] gives us that

$$[P, A] = [P, A, A] \times C_{[P,A]}(A).$$

By [Gor80, 5.3.6] $[P, A, A] = [P, A]$ which forces $C_{[P,A]}(A) = 1$, thereby proving the second statement. □

Lemma 3.12. *Let z_1 and z_2 be involutions in the group G. If $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are strongly closed with $[z_1, z_2] = 1$, then $\langle z_1, z_2 \rangle$ is strongly closed in G as well.*

Proof. Let $z = z_1z_2$, $U = \langle z_1, z_2 \rangle$, and choose $S \in Syl_2(G)$ so that $U \leq S$. Notice that the strong closure of $\langle z_1 \rangle$ and $\langle z_2 \rangle$ forces $U \leq Z(S)$. Now suppose $z^h \in S$ for some $h \in G$. Since $U \leq Z(S)$ we have $U \leq C_G(z^h)$ so then $U^{h^{-1}} \leq C_G(z)$. By Sylow's Theorem, there exists some $k \in C_G(z)$ such that $U^{h^{-1}k} \leq S$ as $S \in Syl_2(C_G(z))$. Since $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are strongly closed $z_i^{h^{-1}k} = z_i$ for both $i = 1, 2$ and hence $z^{h^{-1}k} = z$. This implies $z^h = z$ since $k \in C_G(z)$. Thus we conclude that U is strongly closed as desired. □

The following allows us a useful, equivalent characterization of the notion of weak closure which proves to be independent of a particular Sylow p-group.

Lemma 3.13. *Let G be a group with $S \in Syl_p(G)$. A subgroup P of S is weakly closed in S if and only if P is weakly closed in $N_G(P)$.*

Proof. Suppose that P is weakly closed in S and $P^g \leq N_G(P)$ for some $g \in G$. Clearly $S \in Syl_p(N_G(P))$ so there is some $k \in N_G(P)$ such that $P^{gk} \leq S$. Since P is weakly closed in S we have that $P^{gk} = P$ which forces $P^g = P$ as $k \in N_G(P)$. Suppose on the other hand that P is weakly closed in $N_G(P)$. If $P \trianglelefteq S$, i.e. $S \leq N_G(P)$, then P is obviously weakly closed in S. Otherwise, there is some $s \in N_S(N_S(P)) - N_S(P)$; but this forces $P^s \leq N_G(P)$ which forces $P^s = P$, a contradiction which completes the proof. $\square$

Lemma 3.14. *Let R be a generalized quaternion group of order 2^n where $n > 3$ and let R_0 denote the maximal cyclic subgroup of R. If α is an involution in $\mathrm{Aut}(R)$ such that the α has the modular action on R_0 then $C_R(\alpha)$ is a generalized quaternion group of order 2^{n-1}.*

Proof. Let R have the presentation $R = \langle r, s \mid r^{2^{n-1}} = s^4 = 1, r^{2^{n-2}} = s^2, r^s = r^{-1}\rangle$ where $z = r^{2^{n-2}}$ is the unique involution of R. Then α centralizes r^2 as $r^\alpha = rz$. Since α is an involution either $s^\alpha = s$ or $s^\alpha = zs$. In the latter case, $(rs)^\alpha = rs$ so either $C_R(\alpha) = \langle r^2, s \rangle \cong Q_{2^{n-1}}$ or $C_R(\alpha) = \langle r^2, rs \rangle \cong Q_{2^{n-1}}$. This completes the proof of the lemma. $\square$

Finally, we include a lemma concerning the action of a 3-power element on $Z_{2^n} \times Z_{2^n}$.

Lemma 3.15. *Let $A \cong Z_{2^n} \times Z_{2^n}$ for $n \geq 3$ and take $\alpha \in \mathrm{Aut}(A)$ to be an element of order 3. If $C \leq C_{\mathrm{Aut}(A)}(\alpha)$ is a 2-group then C acts as scalar matrices on A when A is viewed as the free module of rank 2 over $\mathbb{Z}/2^n\mathbb{Z}$ (and $\mathrm{Aut}\, A = GL_2(\mathbb{Z}/2^n\mathbb{Z})$). In particular, C induces the same action on all maximal cyclic subgroups of A, and so C is isomorphic to a subgroup of*

$$(\mathbb{Z}/2^n\mathbb{Z})^\times \cong \mathrm{Aut}(Z_{2^n}) \cong Z_2 \times Z_{2^{n-2}}.$$

Proof. Since α is an automorphism of order 3 on A there is a $\mathbb{Z}/2^n\mathbb{Z}$-basis for which α is represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Direct calculation shows that the only matrices commuting with this are of the form

$$\lambda x + \mu I \text{ for } \lambda, \mu \in \mathbb{Z}/2^n\mathbb{Z}.$$

Since C is a 2-group, it must lie in the subgroup of scalar matrices, as claimed. □

3.4 Facts about $SL_2(q)$ and $L_2(q)$

We record several well-known facts about $SL_2(q)$ and $L_2(q)$ that shall be used throughout this thesis.

Lemma 3.16. *Let $L = SL_2(q)$ for q odd and let $R \in Syl_2(L)$. Then*

(1) R is quaternion.

(2) No involution in $\mathrm{Aut}(L)$ centralizes R.

Proof. Statement (1) can be found in [Gor80, 2.8.3] while statement (2) is in [Foo76a, 2.5]. □

We shall also require the following concerning $L_2(q)$.

Lemma 3.17. *Let $L = L_2(q)$ for $q = p^m$ and p an odd prime, and let $R \in Syl_2(L)$ and $A = \mathrm{Aut}(L)$. Then the following statements hold:*

(1) $A = L\langle i \rangle T$ where i is a PGL-automorphism of L and T is a cyclic group of field automorphisms of order m.

(2) $\mathrm{Out}(A) \cong Z_2 \times Z_m$.

(3) R is dihedral.

(4) All involutions in L are conjugate in L.

(5) A Sylow 2-subgroup of $C_A(R)$ is $Z(R) \times \langle f \rangle$ where f is a field automorphism of order 2 when m is even, and $f = 1$ when m is odd.

Proof. This is [GLS05, 10.1.2] and [Foo76a, 2.5]. □

Definition 3.18. If $U, L \leq G$ with U a four-group normalizing L then we define the subgroup $\Gamma_{U,1}(L) \leq L$ as

$$\Gamma_{U,1}(L) = \langle C_L(w) \mid w \in U^{\sharp} \rangle.$$

Lemma 3.19. *If U is a four-group acting on $L \cong L_2(q)$ and $q > 9$ then $\Gamma_{U,1}(L) = L$.*

Proof. This is [Asc75, 3.6]. □

Chapter 4

Transfer and Preliminary Fusion Systems Lemmas

With the exception of two small lemmas, we dedicate this chapter to developing a fusion systems version of Thompson's Transfer Lemma, a result due to Lynd ([Lyn12]).

Lemma 4.1. *Let $\mathcal{F}$ be a saturated fusion system on a 2-group S and let z_1 and z_2 be involutions in S. If $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are strongly closed (with respect to $\mathcal{F}$) with $[z_1, z_2] = 1$, then $\langle z_1, z_2 \rangle$ is strongly closed as well.*

Proof. Let $z = z_1 z_2$ and $U = \langle z_1, z_2 \rangle$ so that $U \leq Z(S)$. If $z^\phi \in S - U$ we get (by saturation) that ϕ^{-1} extends to a map in $\mathrm{Hom}_\mathcal{F}(C_S(z^\phi), S)$. Since $U \leq C_S(z^\phi)$ we get that $U^{\phi^{-1}}$ is defined. By the strong closure of $\langle z_1 \rangle$ and $\langle z_2 \rangle$ we get that $U^{\phi^{-1}} = U$ and hence $z^{\phi^{-1}} = z$. Thus we conclude that U is strongly closed. □

Lemma 4.2. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Then $O_p(\mathcal{F}) \neq 1$ if and only if S contains a nontrivial, strongly $\mathcal{F}$-closed abelian subgroup.*

Proof. This is [AKO11, I.4.7]. □

4.1 Transfer

We begin by briefly discussing group-theoretic transfer. Let $H \leq G$, $\phi \in \mathrm{Hom}(H, A)$ for an abelian group A, and let $g_1, g_2, ..., g_n$ be a set of right coset representatives of H in G. Then for each $g \in G$ there is a unique $h_i \in H$ such that $g_i g = h_i g_{\sigma_g(i)}$ for all

$1 \leq i \leq n$. Here σ_g is the permutation in S_n where $\sigma_g(i) = j$ if and only if $g_i g$ is in the coset Hg_j. Then the map defined by

$$\tau(g) = \prod_{i=1}^{n} \phi(h_i)$$

is a homomorphism from G into A that does *not* depend on the choice of coset representatives [Gor80, 7.3.2]. This is the *transfer homomorphism* from G into A. If $S \in Syl_p(G)$, for our purposes, the map ϕ will just be projection onto an abelian quotient S/S_0 for some subgroup S_0 of S containing S'. The transfer homomorphism is the principal tool in proving the Focal Subgroup Theorem which describes $S \cap [G, G]$, the focal subgroup of G (relative to a Sylow p-subgroup S). The importance of this result lies in the fact that the maximal abelian p-factor group of G is isomorphic to $S/S \cap [G, G]$.

Theorem 4.3. *If $S \in Syl_p(G)$ then the focal subgroup is given by*

$$S \cap [G, G] = \langle\, x^{-1}x^g \mid x, x^g \in S \textit{ for some } g \in G \,\rangle.$$

Some applications of the Focal Subgroup Theorem, in conjunction with Alperin's Fusion Theorem, include the normal p-complement theorems of Burnside and Frobenius ([Gor80, 7.4.3 and 7.4.5]). For our purposes, however, we will use transfer to prove a fusion systems version of the following generalization of Thompson's Transfer Lemma from [GLS96, 15.15]. An element $u \in S$ is *fully centralized* if $C_S(u) \in Syl_p(C_S(G))$.

Theorem 4.4 (Thompson's Transfer Lemma - group-theoretic version)**.** *Assume the following:*

(a) S contains a proper normal subgroup M such that S/M is cyclic,

(b) $u \in S - M$ *is an element of least order, and*

(c) every fully centralized conjugate of u *is contained in* M *or the coset* Mu*.*

Then either G *has a normal subgroup* N *such that* G/N *is a cyclic* p*-group and* $u \notin N$*, or there exists* $g \in G$ *such that*

(i) $u^g \in M$*,*

(ii) u^g *is fully centralized in* S*, and*

(iii) $C_S(u)^g \leq C_S(u^g)$*.*

We remark that if $p = 2$ then u is an involution and condition (c) holds automatically. Define $O^2(G)$ to be the minimal normal subgroup of G of 2-power index. This admits the following corollary which we apply in Chapter 5.

Corollary 4.5. *Let* $p = 2$ *and suppose* M *is a proper normal subgroup of* S *with* S/M *cyclic. Let* u *be an involution in* $S - M$*. Then either* $O^2(G) < G$ *or there exists* $g \in G$ *such that*

(i) $u^g \in M$*,*

(ii) u^g *is fully centralized in* S*, and*

(iii) $C_S(u)^g \leq C_S(u^g)$*.*

For the ensuing discussion of transfer in fusion systems we follow [AKO11, Section I.8]. In the setting of fusion systems we do not have cosets on which to define a transfer map. Instead, we define a transfer map in terms of a characteristic biset attached to our fusion system $\mathcal{F}$ over a p-group S. For groups G and H a finite set X on which H acts on the left and G acts on the right is a (H, G)-*biset* if these actions commute. We will work with (H, G)-bisets such that the H-left action is free (i.e., $hx = x$

implies $h = 1$), which enables us to take orbit representatives $t_1, ..., t_n$ of the H-left action such that each element of X is written uniquely as $h_i t_i$ for some $1 \leq i \leq n$ and $h_i \in H$. In what follows, these t_i play the role of the coset representatives in the group transfer above.

When X is a (H, G)-biset and Y is a (K, H)-biset we will be interested in forming a new (K, G)-biset as follows. Define $Y \times_H X$ to be the set of orbits under the H-action of $(y, x)h = (yh, h^{-1}x)$ on $Y \times X$. If $[y, x] \in Y \times_H X$ where $[y, x]$ denotes the orbit of (y, x), then we let K and G act by $k[y, x]g = [ky, xg]$. This action is well-defined and turns $Y \times_H X$ into a (K, G)-biset. Also, if $\phi \in \mathrm{Hom}(G, H)$ we will have occasion to form a new (H, G)-biset, $H_{G,\phi}$, with underlying set H. We define the (H, G)-action by letting H act via left multiplication and for $x \in H, g \in G$ we define the G-action on the right by $x \cdot g = xg^{\phi}$.

Denote G/G' by G^{ab}. Then given a (H, G)-biset X whose left H-action is free, we define a map $X_* : G^{\mathrm{ab}} \longrightarrow H^{\mathrm{ab}}$ which will ultimately play the role of our transfer map for a specific biset. Let $\mathcal{T} = \{t_1, ..., t_n\}$ be orbit representatives under the left H-action. Then for $g \in G$ we have unique $h_i \in H$ such that $t_i g = h_i t_{\sigma_g(i)}$ for some $\sigma_g \in S_n$. We define the map $X_*^{\mathcal{T}} : G \longrightarrow H^{\mathrm{ab}}$ by

$$X_*^{\mathcal{T}}(g) = \prod_{i=1}^{n} h_i \bmod H'.$$

It is not difficult to see that $X_*^{\mathcal{T}}$ is independent of the choice of orbit representatives and that $X_*^{\mathcal{T}}$ is a homomorphism. Now let $X_* : G^{\mathrm{ab}} \longrightarrow H^{\mathrm{ab}}$ be the homomorphism induced by $X_*^{\mathcal{T}}$. In [AKO11, I.8.1] the following facts along with a few other useful properties of X_* are proved.

Lemma 4.6. *If X is a (H, G)-biset with free left H-action then the map X_* is a homomorphism from G^{ab} into H^{ab}. Furthermore the following hold.*

(a) If $\phi \in \mathrm{Hom}(G,H)$ *then* $(H_{G,\phi})_* = \phi_*$ *is the homomorphism from* G^{ab} *into* H^{ab} *induced by* ϕ*.*

(b) If $H \leq G$ *and* $X = G$ *is the* (H,G)*-biset defined by letting* H *and* G *act by left and right multiplication respectively, then* X_* *is the group homomorphism* τ_H^G *from* G^{ab} *into* H^{ab} *induced by the ordinary group transfer from* G *into* H^{ab}*.*

(c) If Y *is a* (H,G)*-biset with free left* H*-action then for* $[g] \in G^{ab}$ *we have*

$$(X \amalg Y)_*([g]) = X_*([g]) \cdot Y_*([g]).$$

where $X \amalg Y$ *is the disjoint union of sets* X *and* Y*.*

(d) If Y *is a* (K,H)*-biset with free left action, then*

$$(Y \times_H X)_* = Y_* \circ X_* : G^{ab} \longrightarrow K^{ab}.$$

The next step is to set up the fusion systems framework. For the remainder of this section $\mathcal{F}$ is a saturated fusion system on a p-group S. To construct our transfer map for a saturated fusion system $\mathcal{F}$ over S we need an (S,S)-biset with particular properties.

Definition 4.7. Let $\mathcal{F}$ be a saturated fusion system over S. Then a (S,S)-biset Ω is a *characteristic biset* if it satisfies the following three conditions:

(i) Each (S,S)-orbit has the form $(S_{P,\phi}) \times_P S$ for some $P \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$.

(ii) For each $P \leq S$ and each $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, $\Omega \times_S (S_{P,\mathrm{Id}}) \cong \Omega \times_S (S_{P,\phi})$ as (S,P)-bisets.

(iii) $|\Omega|/|S|$ is prime to p.

The existence of such bisets was initially conjectured by Linckelmann and Webb. The existence of such a biset for any saturated fusion system was later established by Broto, Levi, and Oliver in [BLO03, Proposition 5.5]. Condition (i) ensures that Ω has a left free action, and condition (ii) ensures that right action of P and P^ϕ are the same up to relabeling. In the situation of ordinary transfer and $S \in Syl_p(G)$ we have that $(|G : S|, p) = 1$. Condition (iii) is analogous and will be necessary when proving the fusion system analog of Thompson's Transfer Lemma.

Let $\mathcal{F}$ be a saturated fusion system over S and let Ω be a characteristic (S, S)-biset associated to $\mathcal{F}$. The homomorphism Ω_* will essentially serve as our transfer map from S^{ab} into S^{ab}. First, we decompose Ω by condition (i) above as

$$\Omega = \coprod_{i=1}^{n} (S_{P_i,\phi_i}) \times_{P_i} S \tag{4.1}$$

for some $P_i \leq S$ and $\phi_i \in \mathrm{Hom}_{\mathcal{F}}(P_i, S)$ for $1 \leq i \leq n$. Now we use Lemma 4.6 to determine Ω_*. By parts (a), (b), and (d) we have that

$$((S_{P_i,\phi_i}) \times_{P_i} S)_* = \phi_{i*} \circ \tau^S_{P_i} \quad \text{for each } 1 \leq i \leq n.$$

From part (c) we get that

$$\Omega_* = \prod_{i=1}^{n} \phi_{i*} \circ \tau^S_{P_i}. \tag{4.2}$$

Now we define the focal subgroup of $\mathcal{F}$, the hyperfocal subgroup of $\mathcal{F}$, and $O^p(\mathcal{F})$ by analogy with groups. If $\mathcal{F}$ is a saturated fusion system over S then define the *focal subgroup*, $\mathfrak{foc}(\mathcal{F})$, and *hyperfocal subgroup*, $\mathfrak{hyp}(\mathcal{F})$, by

$$\mathfrak{foc}(\mathcal{F}) = \langle\, g^{-1} g^{\phi} \mid g \in P \leq S, \phi \in \mathrm{Hom}_{\mathcal{F}}(P, S) \,\rangle$$

$$\mathfrak{hyp}(\mathcal{F}) = \langle\, g^{-1}g^{\alpha} \mid g \in P \leq S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\,\rangle.$$

In a saturated fusion system $\mathcal{F}$ on a p-group S we say that a subsystem $\mathcal{E}$ on T has *p-power index* in $\mathcal{F}$ if $T \geq \mathfrak{hyp}(\mathcal{F})$, and $\mathrm{Aut}_{\mathcal{E}}(P) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for each $P \leq S$. Theorem I.7.4 in [AKO11] guarantees the existence of a unique minimal saturated fusion system, $O^p(\mathcal{F})$, that is of p-power index and normal in $\mathcal{F}$. Moreover, $O^p(\mathcal{F})$ is a subsystem on $\mathfrak{hyp}(\mathcal{F}) \trianglelefteq S$.

We have the following important equivalence on $\mathfrak{foc}(\mathcal{F})$, which is in direct analogy with the group situation:

Lemma 4.8. *$\mathfrak{foc}(\mathcal{F}) = S$ if and only if $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{hyp}(\mathcal{F}) = S$.*

Proof. This is [AKO11, I.7.5]. □

In particular, it follows from this lemma that $O^p(\mathcal{F})$ is a *proper* normal subsystem of $\mathcal{F}$ whenever $\mathfrak{foc}(\mathcal{F}) < S$.

Lemma 4.9. *Let $\mathcal{F}$ be a saturated fusion system over S with characteristic biset Ω and transfer map Ω_*. Then $\mathfrak{foc}(\mathcal{F})/S' \leq \ker \Omega_*$.*

Proof. Condition (ii) in the definition of a characteristic biset gives us that $\Omega \times_S S_{P,\mathrm{Id}} \cong \Omega \times_S S_{P,\phi}$ as (S,P)-bisets for $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. It follows that $(\Omega \times_S S_{P,\mathrm{Id}})_* = (\Omega \times_S S_{P,\phi})_*$ as homomorphisms from P^{ab} into S^{ab}. By Lemma 4.6 (a) and (c) $(\Omega \times_S S_{P,\mathrm{Id}})_* = \Omega_* \circ \mathrm{Id}_P = \Omega_*$ and $(\Omega \times_S S_{P,\phi})_* = \Omega_* \circ \phi_*$. Thus

$$\Omega_*([g]) = \Omega_*([g^{\phi}] \quad \text{for } g \in P$$

and we get $[g^{-1}g^{\phi}] \in \ker \Omega_*$, that is, $\mathfrak{foc}(\mathcal{F})/S' \leq \ker \Omega_*$. □

Before proving the Thompson Transfer Lemma for fusion systems we require two results on group transfer.

Lemma 4.10. *Let $H \leq G$ and suppose $x \in Z(G)$. Then $\tau_H^G(x) = x^{|G:H|}H'$.*

Proof. If $g_1, ..., g_n$ are right coset representatives of H in G then $g_i x = x g_i = h_i g_{\sigma_x(i)}$ for some $h_i \in H$ and $\sigma_x \in S_n$. Since $h_i = x g_i g_{\sigma_x(i)}^{-1}$, $x \in Z(G)$, and H^{ab} is abelian we have

$$\tau_H^G(x) = \prod_{i=1}^{n} x g_i g_{\sigma_x(i)}^{-1} H' = \prod_{i=1}^{n} = xH' = x^{|G:H|}H'.$$

□

We also will need the Mackey Decomposition Theorem for transfer.

Lemma 4.11. *Let $H, K \leq G$ and let $H_0 \trianglelefteq H$ with H/H_0 abelian. Let τ be the transfer from G into H/H_0 and for any $t \in G$ let τ_t be the transfer from K into $K \cap H^t / K \cap H_0^t$. If $\mathcal{T}$ is a set of representatives for the H,K-double cosets of G, then for each $k \in K$, we have*

$$\tau(k) = \prod_{t \in \mathcal{T}} t\tau_t(k)t^{-1} \mod H_0.$$

Proof. This is [GLS96, 15.13]. □

We now establish Thompson's Transfer Lemma for fusion systems. We essentially follow Lynd's proof from his doctoral thesis [Lyn12, 3.0.4].

Theorem 4.12 (Thompson's Transfer Lemma — fusion theoretic version)**.** *Suppose M is proper normal subgroup of S with S/M abelian. Let u be an element in $S - M$ of least order. Let $\mathcal{I}$ be the set of fully $\mathcal{F}$-centralized $\mathcal{F}$-conjugates of u in $S - M$, and suppose the set of cosets $\mathcal{I}M = \{vM \mid v \in \mathcal{I}\}$ is linearly independent in $\Omega_1(S/M)$. Then either u has a fully $\mathcal{F}$-centralized $\mathcal{F}$-conjugate in M, or $O^p(\mathcal{F})$ is a proper subsystem of $\mathcal{F}$.*

Proof. Suppose that u has no fully centralized $\mathcal{F}$-conjugate in M. Without loss of generality we suppose that u is fully centralized and let $Q = C_S(u)$. Fix a characteristic biset Ω of $\mathcal{F}$ such that

$$\Omega = \coprod_{i=1}^{n} (S_{M_i,\phi_i}) \times_{M_i} S$$

where $M_i \leq S$ and $\phi_i \in \mathrm{Hom}_{\mathcal{F}}(M_i, S)$ for each i. Let $\tau_{\mathcal{F}}$ be the map from S^{ab} into S/M induced by Ω_*. We shall show that u is not in the kernel of the transfer map. From this it would follow that $\mathfrak{foc}(\mathcal{F}) < S$ and so $O^p(\mathcal{F})$ is a proper, normal subsystem of $\mathcal{F}$.

We now determine $\tau_{\mathcal{F}}(u)$. By the definition of the transfer map and the Mackey Decomposition (Lemma 4.11) we have

$$\begin{aligned} \tau_{\mathcal{F}}(u) &= \prod_{i=1}^{n} (\phi_{i*} \circ \tau_{M_i}^{S})(u) \\ &= \prod_{i=1}^{n} \prod_{t \in [M_i \backslash S/Q]} \phi_{i*}(t\tau_t(u)t^{-1}) \end{aligned}$$

where $\tau_t = \tau^{Q}_{M_i^t \cap Q}$. Since $u \in Z(Q)$ it follows from Lemma 4.10 that $\tau_t(u) = u^{|Q:Q \cap M_i^t|}$. Therefore we have

$$\tau_{\mathcal{F}}(u) = \prod_{i=1}^{n} \prod_{t \in [M_i \backslash S/Q]} \phi_{i*}(tut^{-1})^{|Q:M_i^t \cap Q|}.$$

If $|Q : M_i^t \cap Q|$ is divisible by p then since u is of least order in $S - M$ we have that $\phi_{i*}(tut^{-1})^{|Q:M_i^t \cap Q|} \in M$; this factor then contributes nothing to the transfer. On the other hand, $|Q : M_i^t \cap Q| = 1$ if and only if $Q \leq M_i^t$. Since u is fully centralized and ϕ_i is defined on $Q^{t^{-1}} = C_S(tut^{-1}) \leq M_i$, it follows that $\phi_i(tut^{-1})$ is fully centralized as well. By assumption we have that $\phi_i(tut^{-1}) \notin M$ and therefore contributes to the transfer. Let l equal the number of pairs (i,t) such that $Q \leq M_i^t$. Then by the above remarks,

$$\tau_{\mathcal{F}}(u) = \prod_{\overline{v} \in \mathcal{I}M} v^{k_v} \mod M$$

with $\sum_{\overline{v} \in \mathcal{I}M} k_v = l$.

Now we shall show that $(l,p) = 1$. Observe that Q fixes the coset $M_i t$ in its right action if and only if $Q \leq M_i^t$. So we decompose Ω into disjoint orbits of the form $rM_i t$ where r is a coset representative in Ω, $1 \leq i \leq n$, and $t \in [M_i \backslash S/Q]$. It follows that l is the number of fixed points of Q on this set of orbits. Condition (iii) on characteristic bisets gives us that $(|\Omega|/|S|, p) = 1$ and hence the number of orbits is relatively prime to p. Since Q is a p-group, the number of fixed points of Q on this set is equal to the total number of orbits mod p, that is, the number of fixed points is relatively prime to p.

Since $(l,p) = 1$ it follows that $p \nmid k_v$ for some $\overline{v} \in \mathcal{I}M$. By the linear independence of $\mathcal{I}M$, we have that

$$\tau_{\mathcal{F}}(u) = \prod_{\overline{v} \in \mathcal{I}M} v^{k_v} \mod M \neq 1 \mod M.$$

We have shown that $u \notin \ker \tau_{\mathcal{F}}$ which completes the proof. □

We remark that when $p = 2$ and S/M is cyclic the linear independence condition is automatically satisfied. We obtain the following corollary which applies to our situation in Chapter 6.

Corollary 4.13. *Let $p = 2$ and suppose M is a proper normal subgroup of S with S/M cyclic. Let u be an involution in $S - M$. Then either u has a fully $\mathcal{F}$-centralized $\mathcal{F}$-conjugate in P or $O^2(\mathcal{F})$ is a proper subsystem of $\mathcal{F}$.*

Chapter 5

The Group Theoretic Proof for $SL_2(q)$

In this chapter we consider a fusion simple group G with an involution t such that $C_G(t)$ has a standard component L of type $SL_2(q)$ for q odd. Let $S \in Syl_2(G)$ and $R \in Syl_2(L)$. In Section 1.3 we showed that we can assume that R is a weakly closed generalized quaternion subgroup which is strongly closed in $C_G(y)$ for every $y \in C_S(R)$. Under this weaker hypothesis, we will show that S has a proper strongly closed subgroup or that S is wreathed or of $G_2(q)$-type. This actual standard form problem for $SL_2(q)$ is established in [Foo76a, Foo76b] but unlike the original group-theoretic proof, our analysis is strictly 2-local and fusion-theoretic, that is, we do not "see" the whole component L, only its Sylow 2-subgroup, R. (Our arguments are also similar, but not identical to those in [Asc77b, Section 19]) This makes our reduced hypothesis well-suited to the context of fusion systems. Our proof in this chapter provides a partial template for proving a fusion systems analog of the theorem in the next chapter. The main theorem of this chapter is the following.

Theorem 5.1. *Let G be a finite group with $O^2(G) = G$, $S \in Syl_2(G)$, and R a generalized quaternion subgroup of S with unique involution z. Assume further that*

(1) R is weakly closed in S with respect to G and

(2) R is strongly closed in S with respect to $C_G(y)$ for every involution $y \in C_S(R)$.

Then either S is of $G_2(q)$-type, wreathed or S contains a strongly closed subgroup of one of the following types:

(i) $\langle z \rangle$,

(ii) $\langle z, z^g \rangle$ *for some* $g \in G$*, a four-group, or*

(iii) a quasidihedral group.

In fact, we will show even more about S: first, we suppose $|R| = 2^n$. Then in the case S is of $G_2(q)$-type it will follow that $S \cong (Q_1 * Q_2)\langle t \rangle$ where $Q_1 \cong Q_2 \cong Q_{2^n}$ and $R = Q_1$ or Q_2 so that $|S| = 2^{2n}$. When S contains a strongly closed quasidihedral group P we will have that $P = R\langle z^g \rangle$ for an appropriate $g \in G$ so that $|P| = 2^{n+1}$. Finally, when S is wreathed we will have that $S \cong Z_{2^{n-1}} \wr Z_2$ so that $|S| = 2^{2n-1}$.

We begin by considering the unique involution z of R. We notice immediately that since R is weakly closed in S, $R \trianglelefteq S$, and since $\langle z \rangle \operatorname{char} R \trianglelefteq S$ we get that $\langle z \rangle \trianglelefteq S$. This forces $z \in Z(S)$ and so $S \in Syl_2(C_G(z))$.

For the remainder of this chapter we will suppose $\langle z \rangle$ is not strongly closed and hence there is some $g \in G$ such that $z \neq z^g \in S$. This forces $[z, z^g] = 1$ by the above remarks. Our proof will very much depend on the structure of certain subgroups which we initially take to be

$$R_0 = C_R(z^g) \quad \text{and} \quad T = C_{R^g}(z).$$

We next modify $g \in G$ so that the additional property $T \leq S$ holds; we show that such an element $g \in G$ exists. Observe that since $T \leq C_G(z)$ we can choose $k \in C_G(z)$ such that $T^k = C_{R^{gk}}(z) \leq S$ and we replace g by gk. Notice that z^{gk} may differ from z^g so R_0 now become $C_R(z^{gk})$. Finally, *of all such* g *having the properties* $z^g \in S$ *and* $T = C_{R^g}(z) \leq S$*, we choose* g *such that* $|T|$ *is maximal.*

We fix this notation and claim that $|R_0| \leq |T|$. Observe that since $R_0 \leq C_G(z^g)$, there is some $c \in C_G(z^g)$ such that $R_0 \leq S^{gc}$. We replace g by gc and notice that this leaves R_0 unchanged as $z^{gc} = z^g$. It then follows that

$$R_0^{g^{-1}} = C_{R^{g^{-1}}}(z) \leq S.$$

As $z^{g^{-1}} \neq z$, by the maximality of T we get

$$|R_0| = |R_0^{g^{-1}}| \leq |T|$$

which completes the proof of the claim.

The first step in our proof of Theorem 5.1 is to show that R cannot commute with z^g.

Lemma 5.2. $[z^g, R] \neq 1$.

Proof. If $[z^g, R] = 1$ then $R \leq C_G(z^g)$ so there is some $h \in C_G(z^g)$ such that $R^h \leq S^g$ and hence by weak closure of R in S, $R^h = R^g$. It follows then that $R^{gh^{-1}} = R$ and $z^{gh^{-1}} = z$. But $h^{-1} \in C_G(z^g)$ so $z^{gh^{-1}} = z^g = z$, a contradiction. □

Next we show that R_0 and T commute.

Lemma 5.3. *R_0 and T centralize each other, i.e.* $[R_0, T] \leq R_0 \cap T = 1$.

Proof. Note that we have chosen g in such a way that T normalizes R, so T normalizes $C_R(z^g) = R_0$. Thus $[R_0, T] \leq R_0$. Since $TR_0 \leq C_G(z^g)$, by Sylow's Theorem there is some $h \in C_G(z^g)$ such that $(TR_0)^h \leq S^g$. Since R is strongly closed in $C_G(z)$, we have that

$$T^h \leq R^g.$$

Now, $R^g \trianglelefteq S^g$ so $[R_0^h, T^h] \leq R^g$; but above we saw that $[R_0^h, T^h] \leq R_0^h$. So we have that

$$[R_0^h, T^h] \leq R^g \cap R_0^h.$$

If $R^g \cap R_0^h \neq 1$ then it follows that $z^g = z^h$ (being the unique involutions of R^g and R_0 respectively). This implies that

$$z^g = z^{gh^{-1}} = z$$

which is a contradiction. This shows $1 = [R_0^h, T^h] = [R_0, T]^h$, as needed.

□

Lemma 5.4. *$R_0 = \langle z \rangle$ or R_0 is a maximal cyclic subgroup of R (which is unique for $n > 3$).*

Proof. Assume $R_0 \neq \langle z \rangle$ and let $N = N_{TR}(T)$. Suppose first that $N = TR$. In this case

$$[T, R] \leq T \cap R = 1$$

as T and R have distinct, unique involutions. From this it follows that $[z^g, R] = 1$, a contradiction by Lemma 5.2. This shows that $N \neq TR$.

Since TR is a 2-group $N < N_{TR}(N)$, and so we can choose an element $s \in R$ such that

$$s \in N_{TR}(N) - N \text{ and } T^s \neq T.$$

We see that T and T^s normalize each other as

$$N_{TR}(T) = N_{TR}(T)^s = N_{TR}(T^s).$$

It follows then that $[T, T^s] \leq T \cap T^s$. If $z^g \in T \cap T^s$ then it follows that $z^{gs} = z^g$ and so $s \in C_G(z^g)$. Then we have that $s \in R_0$; but R_0 normalizes T by Lemma 5.3 so

$z^g \neq z^{gs}$, i.e., T and T^s have distinct, unique involutions. Thus $[T^s, T] \leq T^s \cap T = 1$, i.e., $TT^s = T \times T^s$.

Now notice that

$$T \times T^s \leq C_{TR}(z^g) = TR_0.$$

Since $|R_0| \leq |T|$ this forces

$$T \times T^s = TR_0 \text{ and } T \cong R_0.$$

Next, we claim that $|TR : TR_0| = 2$. If $R_0 = R$ this means that $[z^g, R] = 1$ which is a contradiction by Lemma 5.2 so $|TR : TR_0| \geq 2$. Now let

$$W = \Omega_1(TR_0) = \langle z, z^g \rangle.$$

Since W is a four-group we have that $\mathrm{Aut}(W) \cong S_3$. Notice also that $C_{TR}(W) = TR_0$ and

$$W \operatorname{char} TR_0 \trianglelefteq N_{TR}(TR_0) \leq N_{TR}(W).$$

Now $N_{TR}(W)$ acts on W and so

$$1 \neq N_{TR}(W)/C_{TR}(W) = N_{TR}(W)/TR_0 \leq S_3.$$

It then follows by order considerations that $|N_{TR}(W) : TR_0| = 2$. Consequently, all squares in $N_{TR}(W)$ are contained in TR_0. If $N_{TR}(W) \neq TR$ then it follows that there is some $u \in N_{TR}(N_{TR}(W)) - N_{TR}(W)$. Since $|T| \geq 4$ by assumption, z^g is a square in T. Hence $(z^g)^u$ is a square in $N_{TR}(W)$ and we have that $z^{gu} \in TR_0$. Since $z^u = z$

we have that $W^u = W$, i.e. $u \in N_{TR}(W)$, a contradiction. Therefore it follows that

$$N_{TR}(W) = TR \text{ and so } |TR : TR_0| = 2$$

as claimed. It remains to show that R_0 is maximal cyclic.

Since R is generalized quaternion all subgroups are cyclic or quaternion. But notice that $[T, R_0] = 1$ and $[T, T^s] = 1$. Since

$$T \times T^s = T \times R_0 = T^s \times R_0$$

we have that $T \leq Z(T \times T^s)$ which gives us that T is abelian and hence cyclic. Since $|R : R_0| = 2$ we have that $R_0 \cong T$ is a maximal cyclic subgroup of R.

□

At this point we observe that for $n = 3$, R has three maximal cyclic subgroups. Since S is a 2-group at least one of these maximal subgroups M must be S-invariant. In the case R_0 is a maximal cyclic subgroup, z^g acts non-trivially on R (by Lemma 5.2), centralizes R_0, and normalizes M. But T normalizes M as well and z^g is a square in T which gives us that z^g centralizes M. This all forces $R_0 = M$ and, by similar arguments, T is S^g-invariant as well. These facts will be useful throughout the remainder of the chapter, and we assume them whenever R_0 is a maximal cyclic subgroup of R.

Lemma 5.5. *For any two distinct commuting conjugates z_1, z_2 of z, z_1z_2 is also conjugate in G to z.*

Proof. We may assume $z_1 = z$ and $z_2 = z^h \in S$. Let $X = R\langle z^h \rangle$, $U = \langle z, z^h \rangle$, and $C = C_X(z^h) = C_X(U)$. First, we observe that $R \not\leq C$ by Lemma 5.2 and hence $C < N_X(C)$. Now we claim U is the unique four-group in C. If there were another four-group $U_1 \neq U$ then $\langle U, U_1 \rangle$ would contain an E_8 which does not happen in X since R is of rank 1. We have then that $U \operatorname{char} C$ and therefore $N_X(C)$ normalizes U. It follows at once that for any $y \in N_X(C) - C$ we get $(z^h)^y = z^{hy} = zz^h$ as desired. □

We now prove Theorem 5.1 in the case where $|T| = 2$, or equivalently where $C_R(z^h) = \langle z \rangle$ whenever $z^h \in S - \langle z \rangle$ for any $h \in G$.

Lemma 5.6. *Suppose $C_R(z^h) = \langle z \rangle$ whenever $z^h \in S - \langle z \rangle$ for any $h \in G$. Then $R\langle z^g \rangle$ is quasidihedral and strongly closed in S with respect to G.*

Proof. First, we have that $P = R\langle z^g \rangle$ is quasidihedral by Suzuki's Lemma. Next, we show that P is *strongly involution closed*, that is, the set of involutions in P is strongly closed. Since P is quasidihedral there are only two P-classes of involutions: one consisting of just z (a central element) and $(z^g)^P$. Thus if P is not strongly involution closed there is some

$$t = z^h \in S - P.$$

Adjusting h by Sylow's Theorem (in $C_G(z^h)$ if necessary) we can assume that

$$C_P(t) \leq S^h.$$

Therefore $C_P(t)$ acts on R^h by weak closure. We claim that $C_P(t) = \langle z \rangle$. Suppose not, that is, $|C_P(t)| \geq 4$. But R^h has an S^h-invariant subgroup $H \cong Z_4$ on which $C_P(t)$ acts. Since $\operatorname{Aut}(Z_4) \cong Z_2$ it follows that the squares in $C_P(t)$ centralize H. It follows then that some involution in $C_P(t)$ is centralized by H. Since all involutions in P are conjugate to z in G we have that $|C_{R^h}(z^k)| \geq 4$ for some k in G. Restated, we have shown that $|C_R(z^{kh^{-1}})| \geq 4$, a contradiction. Thus we conclude that

$$C_P(t) = \langle z \rangle.$$

By Suzuki's Lemma $P\langle\, t\,\rangle$ is quasidihedral. Note that $P\langle\, t\,\rangle$ contains the quasidihedral subgroup P. This is impossible as the only proper subgroups of a quasidihedral group are cyclic, dihedral, or quaternion. We therefore conclude that P is strongly involution closed.

If P is not strongly closed then there is some $x \in P$ such that $x^k \in S - P$, for some $k \in G$. By the above arguments, x is not an involution. Some power of x is the involution $z \in P$ and hence $z^k \in P$ as P is strongly involution closed. First suppose that $z^k \neq z$. Since x^k normalizes R we have, by the arguments above, that z^k centralizes a cyclic group of order 4 in R thereby contradicting our assumption that $C_R(z^k) = \langle z \rangle$. So we have that $z^k = z$, i.e. $k \in C_G(z)$.

Since P is quasidihedral we have that its three maximal subgroups (of index 2) are M_1 (cyclic), M_2 (dihedral), and R (generalized quaternion). This gives us that $\Phi(P)$ is the maximal cyclic subgroup of R and we have $|P : \Phi(P)| = 4$ by Lemma 3.7. Then it follows that

$$P/\Phi(P) \cong M_2/\Phi(P) \times R/\Phi(P) \cong \langle \overline{z_1} \rangle \times \langle \overline{z_2} \rangle$$

where we choose z_1 to be an involution in M_2. Because by our overall hypotheses R is strongly closed in $C_G(z)$ we have that $x \notin R$. If $x \in M_2$ then since x is not an involution we have $x \in \Phi(P) \leq R$ which again, cannot be the case. Thus we have $x \in M_1$. Since $x \notin R$ and $|R : \Phi(P)| = 2$ we have further that $\langle x \rangle = M_1$. Since

$$\langle x^2 \rangle = \Phi(P) \leq R$$

we have that $(x^k)^2 \in R$ (again by strong closure) and so

$$\Phi(P) = \langle (x^k)^2 \rangle \text{ i.e., } k \in N_G(\Phi(P)) \leq C_G(z).$$

Since R is strongly closed (with respect to $N_G(\Phi(P))$) we have that $R/\Phi(P)$ is strongly closed in $S/\Phi(P)$ with respect to $N_G(\Phi(P))/\Phi(P)$. If $z_1^k \in S$ then $z_1^k \in P$ as P is strongly involution closed. As the only involution of M_1 and R is z, it follows immediately that $z_1^k \in M_2$, i.e., $\overline{z_1^k} \in \langle \overline{z_1} \rangle$ and we get that $\langle \overline{z_1} \rangle$ is strongly closed in $S/\Phi(P)$ with respect to $N_G(\Phi(P))/\Phi(P)$. By Lemma 3.12 we have that $P/\Phi(P) \cong \langle \overline{z_1} \rangle \times \langle \overline{z_2} \rangle$ is strongly closed as well. Hence P is strongly closed in S with respect to $N_G(\Phi(P))$, contrary to the existence of $x^k \in S - P$, $k \in N_G(\Phi(P))$.

□

To complete the proof of Theorem 5.1 we now consider when $R_0 = C_R(z^g)$ is a maximal cyclic subgroup of R.

Lemma 5.7. *$P = TR \cong Z_{2^{n-1}} \wr Z_2$.*

Proof. By the proof of Lemma 5.4 we have that $TR_0 = T \times T^s = \langle a \rangle \times \langle b \rangle$ where $\langle\, a \,\rangle = T$ and $b = a^s$ for $s \in R - R_0$. Then since $s \notin R_0$ we have that

$$b^s = (a^s)^s = a^z = a$$

so s interchanges a and b by conjugation.

Since R is normal in P we have that

$$a^{-1}b = a^{-1}a^s = [a, s] \in R$$

is an element of order 2^{n-1} and so $R_0 = \langle a^{-1}b \rangle$. Now $t = a^{2^{n-2}}s = z^g s$ is a wreathing involution so that we now have

$$P = TR_0\langle s\rangle = (\langle a\rangle \times \langle b\rangle) \rtimes \langle t\rangle \cong Z_{2^{n-1}} \wr Z_2.$$

□

For the rest of this chapter we shall maintain the above notation for P.

Lemma 5.8. *$W = \langle\, z, z^g \,\rangle$ is normal in S, and so $|S : C_S(z^g)| = 2$.*

Proof. Let $A = TR_0$ so if W is not normal in S then neither is A, as $W \operatorname{char} A$. Suppose this is the case and take $u \in N_S(N_S(A)) - N_S(A)$ so

$$A^u \leq N_S(A) = N_S(A^u) = N_S(A)^u.$$

We first suppose that $W^u = W$. This forces $z^g \in W^u$ so that $z^{gu} = z^g$ or $z^{gu} = zz^g$. Since $TR = A\langle\, t\,\rangle \cong Z_{2^{n-1}} \wr Z_2$ we replace u by ut if necessary (which still does not normalize A) to get $z^{gu} = z^g$ so $u \in C_G(z^g)$. Since $\langle\, T, R_0, u\,\rangle \leq S$ is a 2-group contained in $C_G(z^g)$ we can choose $k \in C_G(z^g)$ such that

$$\langle\, T, R_0, u\,\rangle^k = \langle\, T^k, R_0^k, u^k\,\rangle \leq S^g.$$

But R^g is strongly closed in $C_G(z^g)$ which forces $T^k = T$. Since u normalizes R_0 and u^k normalizes T it follows then that u^k normalizes $T \times R_0^k$ and hence u normalizes $A = T \times R_0$ contrary to assumption. This shows that $W^u \neq W$.

As A and A^u normalize each other, we get that $[A, A^u] \leq A \cap A^u \leq R_0$. But then

$$T \leq N_S(A) = N_S(A^u) \leq N_S(W^u).$$

This forces $[z^{gu}, T] \le W^u$ and hence

$$[z^{gu}, T] \le R_0 \cap W^u = \langle z \rangle.$$

Now let $X = \langle z^{gu}, z, T \rangle$ so that

$$\overline{X} = X/\langle z \rangle \cong Z_2 \times Z_{2^{n-1}}.$$

By Lemma 3.6 we have that $\overline{X}$ cannot act faithfully on R. Thus at least one of $\overline{z^g}, \overline{z^{gu}}$, or $\overline{z^g z^{gu}}$ centralizes R. We see from Lemma 5.2 that $z^G \cap C_G(R) = \{z\}$ and hence $z^g, z^{gu} \notin C_G(R)$. Since z^g is a square in T which acts on W^u we have that $[z^g, z^{gu}] = 1$ and so by Lemma 5.5 we have that $z^g z^{gu}$ is conjugate to z; this contradicts the fact that $z^G \cap C_G(R) = \{z\}$. Since S centralizes z it follows by order considerations that $|S : C_S(W)| = |S : C_S(z^g)| = 2$, completing the proof. □

Now that we know $W \trianglelefteq S$ and $C_S(W) = C_S(z^g)$ is of index 2 in S, we may assume (possibly after conjugation in $C_G(z^g)$) that $C_S(z^g) = S \cap S^g$. The following lemmas determine the structure of $S \cap S^g$.

Lemma 5.9. *Let $A = TR_0 \cong Z_{2^{n-1}} \times Z_{2^{n-1}}$. Then there is an $x \in N_G(A)$ of 3-power order such that*

$$S \cap S^g = A \rtimes C_{S \cap S^g}(\langle x \rangle).$$

Proof. First we find the required $x \in G$. If we take $s \in R - R_0$ and $s_1 \in R^g - T$ we get $(z^g)^s = zz^g$ and $z^{s_1} = zz^g$, i.e., $s_1 s$ normalizes W and induces an automorphism of order 3 on it. Recall $t = z^g s$ and now let $t_1 = zs_1$ so then $x_1 = t_1 t$ acts as the same automorphism on W but in this case t, t_1 are both involutions and so $\langle t, t_1 \rangle$ is a dihedral group. As an automorphism of W, x_1 has order 3 and as $\langle x_1 \rangle$ is the (unique) maximal cyclic subgroup of $\langle t, t_1 \rangle$, we choose x an element of 3-power order in $\langle x_1 \rangle$ inducing the same automorphism x_1 on W. Because s and s_1 act on $S \cap S^g$, so too does x. Since this is a $2'$-action, Fitting's Lemma (Lemma 3.11) gives us that

$$S \cap S^g = [S \cap S^g, x]C \quad \text{where} \quad C = C_{S \cap S^g}(\langle x \rangle) \text{ and } t \text{ inverts } \langle x \rangle.$$

As R and R^g are normal in S and S^g respectively, we know that

$$[S \cap S^g, t] = [S \cap S^g, s] \leq R \cap S^g = R_0$$

and similarly, $[S \cap S^g, t_1] \leq T$. Now we have that $[S \cap S^g, x] = [S \cap S^g, tt_1] \leq A$. Since $[S \cap S^g, x]$ is abelian, Lemma 3.11 gives us $[S \cap S^g, x] \cap C = 1$ which forces $[S \cap S^g, x] = A$. We have therefore shown that $S \cap S^g = A \rtimes C$ as desired. □

We maintain the notation of Lemma 5.9 above for the remainder of the chapter. We point out for future reference that since s inverts x it follows that s normalizes C. But since R is normal in S we also get that $[s, C] \leq R \cap C = 1$.

We record another important consequence of the preceding lemma.

Lemma 5.10. *C acts faithfully on A, and so $C_S(A) = A$. Moreover, C induces the same action on all maximal cyclic subgroups of A and so C is isomorphic to a subgroup of*

$$(\mathbb{Z}/2^{n-1}\mathbb{Z})^\times \cong \mathrm{Aut}(Z_{2^{n-1}}) \cong Z_2 \times Z_{2^{n-3}}.$$

Proof. We saw in Lemma 5.9 that $S \cap S^g = A \rtimes C$ and so it follows that

$$S = (A \rtimes C)\langle t \rangle.$$

But then $C^* = C_S(A) \cap C \trianglelefteq S$ because it is normalized by A, C, and t. If $C^* \neq 1$ then since it is a 2-group there is some involution $u \in C^* \cap Z(S)$. By our overall hypotheses

R is strongly closed in $C_G(u)$. Since $u \in C$, $u^x = u$; and since $x \in C_G(u)$ but $\langle z, z^x \rangle$ is a 2-group, $z^x = z$ by strong closure, a contradiction as $z^x = z^g$.

The second assertion is just Lemma 3.15 $\square$

Next, we will show in the case $R \cong Q_8$ that S is completely determined.

Theorem 5.11. *If $R \cong Q_8$ then $S \cong (Q_8 * Q_8)\langle t \rangle$ is of $G_2(q)$-type or $S = TR \cong Z_4 \wr Z_2$ is wreathed.*

Proof. So far we have established that $S = (A \rtimes C)\langle t \rangle$. By the previous lemma $C \leq \mathrm{Aut}(Z_4) \cong Z_2$ which forces $C = \langle r \rangle$ where $r = 1$ or r acts by inversion on A. In the former case we have at once that $S = A\langle t \rangle \cong Z_4 \wr Z_2$. Now $[t, C] = 1$ as C commutes with s and z^g so in the latter case $S = A\langle r, t \rangle$ is of $G_2(q)$-type by Lemma 3.9. $\square$

Now we move on to the case $n \geq 4$.

Theorem 5.12. *If $n \geq 4$ and W is not strongly closed then one of the following holds:*

(i) $S = P \cong Z_{2^{n-1}} \wr Z_2$ is wreathed, or

(ii) $S \cong (Q_{2^n} * Q_{2^n})\langle t \rangle$ is of $G_2(q)$-type.

Proof. We have that $n \geq 4$ so

$$S = A(C\langle t \rangle)$$

where $C \leq Z_2 \times Z_{2^m}$ for some $m \geq 0$. By assumption, W is not strongly closed and so there is some G-conjugate z^h in $S - A$. Let $\overline{S} = S/A$ so that $\overline{S} \cong C\langle t \rangle$. Clearly $\overline{z^h} \neq \overline{1}$. We first establish that if $\overline{z^h} \in \overline{C}$ then z^h cannot act on the maximal cyclic subgroups of R_0 as the modular automorphism. Suppose the contrary. Then by Lemma 3.14 we have $C_R(z^h) \cong Q_{2^{n-1}}$. By Sylow's Theorem we may choose $k \in C_G(z^h)$ such that

$(C_R(z^h))^k = C_{R^k}(z^h) \leq S^h$. Since $[z^k, R^h] \neq 1$ by Lemma 5.2, order considerations give that $|C_{R^k}(z^h)| \geq |C_{R^h}(z^k)|$. Letting $C_{R^h}(z^k)$ play the role of R_0 and $C_{R^k}(z^h)$ play the role of T, we invoke Lemmas 5.2-5.4 to get that $C_{R^k}(z^h)$ is maximal cyclic in R^k, a contradiction.

Now we wish to show that $C = \langle\, r \,\rangle$ where $r = 1$ or r is a non-modular involution acting on A. Suppose not so there exists d a modular involution in C that centralizes the subgroup of index 2 in R_0. Moreover, $C = \langle\, i \,\rangle \times \langle\, e \,\rangle$ where either $i = 1$ or i acts by inversion on A, and where d is the unique involution in $\langle\, e \,\rangle$. Then notice that in this case

$$C_S(d) = (\langle\, a^2 \,\rangle \times \langle\, b^2 \,\rangle)(C\langle\, t \,\rangle) \tag{5.1}$$

which is of index 4 in S. Let $M = A\langle\, i \,\rangle\langle\, t \,\rangle$ so $S/M \cong Z_{2^m}$. By the Thompson Transfer Lemma (Lemma 4.5) we can choose $k \in G$ such that

$$d^k \in M \text{ and } C_S(d)^k \leq C_S(d^k). \tag{5.2}$$

If $d^k \in W$ then d is G-conjugate to z; this gives a contradiction by the observation above that any G-conjugate of z in C must convert the maximal cyclic subgroups of A into dihedral or quasidihedral groups. Thus $d^k \notin A$. We must have then that $\overline{d^k} \in \langle\, \bar{i} \,\rangle \times \langle\, \bar{t} \,\rangle$ in $\overline{S} = S/A$. If $\overline{d^k} = \bar{i}$ then

$$C_S(d^k) \leq W(C\langle\, t \,\rangle)$$

which is of index $2^{n-2}2^{n-2} > 4$ in S. If $\overline{d^k} = \bar{t}$ then

$$C_S(d^k) \leq \langle\, ab \,\rangle(C\langle\, t \,\rangle)$$

which is of index $2^{n-1} > 4$ in S. Finally, if $\overline{d^k} = \overline{it}$ then

$$C_S(d^k) \leq \langle\, ab^{-1} \,\rangle (C\langle\, t \,\rangle)$$

which is of index $2^{n-1} > 4$ in S. Whatever the case, we get $|C_S(d)^k| > |C_S(d^k)|$ which is a contradiction to (5.2). Thus $C = \langle\, r \,\rangle$ and

$$S = A\langle\, r \,\rangle\langle\, t \,\rangle.$$

If $r = 1$ we get $P = S \cong Z_{2^{n-1}} \wr Z_2$. If r inverts A then by Lemma 3.9, $S \cong (Q_{2^n} * Q_{2^n})\langle\, t \,\rangle$.

It remains to consider when r has the quasidihedral action on A. Since W is not strongly closed, we have $z^h \in S - A$ for some $h \in G$. Now because r has the quasidihedral action on R_0 and t inverts R_0, rt acts modularly on R_0. Therefore z^h must equal r or t (mod A) by the argument in the first paragraph of this proof. We apply Thompson's Transfer Lemma again to S/M where $M = A\langle\, r \,\rangle$ or $M = A\langle\, t \,\rangle$ to get that all involutions in S are conjugate to z. But then rt is conjugate to z which again gives a contradiction by the same argument in the first paragraph of this proof. This completes the proof of Theorem 5.1.

□

Chapter 6

The Proof of Theorem 1

In this chapter we will "translate" the group theoretic classification of the previous chapter into strict fusion systems language. In most instances, the proofs will be identical or at least analogous to those of the previous chapter. In this chapter we consider a saturated fusion system $\mathcal{F}$ on a 2-group S such that $O^2(\mathcal{F}) = \mathcal{F}$ and $O_2(\mathcal{F}) = 1$. We assume further that S contains a weakly closed generalized quaternion subgroup R and for every involution $y \in C_S(R)$, R is strongly closed in $C_{\mathcal{F}}(y)$. Under these hypotheses we prove the following theorem for "standard" components of type $SL_2(q)$. See Chapters 1 and 2 for a discussion of our motivation.

Theorem 6.1. *Let $\mathcal{F}$ be a saturated fusion system on a 2-group S such that $O_2(\mathcal{F}) = 1$ and $O^2(\mathcal{F}) = \mathcal{F}$. Suppose R is a generalized quaternion subgroup of S. Assume further that*

(1) R is weakly closed in S with respect to $\mathcal{F}$,

(2) R is strongly closed in $C_{\mathcal{F}}(\langle y \rangle)$ for each involution $y \in C_S(R)$.

Then $\mathcal{F}$ is the fusion system of $L_3(q)$ or $G_2(q)$, for some q odd or S contains a strongly $\mathcal{F}$-closed quasidihedral subgroup.

In fact, we show even more about $\mathcal{F}$: first, we suppose $|R| = 2^n$. Then in the case S is of $G_2(q)$-type it will follow that $S \cong (Q_1 * Q_2)\langle t \rangle$ where $Q_1 \cong Q_2 \cong Q_{2^n}$ and $R = Q_1$ or Q_2 so that $|S| = 2^{2n}$. When $\mathcal{F}$ is the fusion system of $L_3(q)$, S is wreathed with $S \cong Z_{2^{n-1}} \wr Z_2$ so that $|S| = 2^{2n-1}$. Finally, when S contains a strongly closed quasidihedral group P we will show that $P = R\langle z^\phi \rangle$ for an appropriate $\phi \in \mathrm{Hom}_\mathcal{F}(\langle z \rangle, S)$ and $\langle z \rangle = Z(R)$, so that $|P| = 2^{n+1}$.

Since we do not have Sylow's Theorem (or an ambient group at all), our proof of Theorem 6.1 is going to hinge on our ability to exploit the "Sylow-like" axioms of *saturated* fusion systems. In particular, we will make frequent use of the axiom allowing us to extend a map $\phi \in \mathrm{Iso}_\mathcal{F}(P, Q)$ up to N_ϕ, where Q is a fully centralized subgroup. We recall that

$$N_\phi = \{g \in N_S(P) \mid \phi^{-1} c_g \phi \in \mathrm{Aut}_S(Q)\}.$$

Also we use repeatedly without mention that if $\phi \in \mathrm{Hom}_\mathcal{F}(P, Q)$ and $s \in S$ then $(P^\phi)^s = P^{\phi c_s}$ is well-defined (see Section 2.1).

In the previous chapter we saw that $z \in Z(S)$ and $R \trianglelefteq S$ which is easily seen to hold here as well. It follows that $\langle z \rangle$ is fully centralized in $\mathcal{F}$. If $\phi \in \mathrm{Hom}_\mathcal{F}(\langle z \rangle, S)$ then since $\mathcal{F}$ is saturated and $\langle z \rangle = \langle z^\phi \rangle^{\phi^{-1}}$ is fully centralized, we have that ϕ^{-1} extends from $\langle z^\phi \rangle$ to a map (also denoted by ϕ^{-1})

$$\phi^{-1} : C_S(z^\phi) \to S$$

as $C_S(z^\phi) \leq N_{\phi^{-1}}$. We will make frequent use of such extensions.

As an immediate consequence of weak closure, we first observe that no distinct $\mathcal{F}$-conjugate of z commutes with R.

Lemma 6.2. *If $z^\phi \in S - R$ for some $\phi \in \mathrm{Hom}_\mathcal{F}(\langle z \rangle, S)$ then $[z^\phi, R] \neq 1$.*

Proof. If $[z^\phi, R] = 1$ then $R \leq C_S(z^\phi)$. Since $\mathcal{F}$ is saturated, we can extend ϕ^{-1} to

$R \le C_S(z^\phi)$ and so $R^{\phi^{-1}} \le S$. Since R is weakly closed in $\mathcal{F}$ we get that

$$R^{\phi^{-1}} = R \text{ and } z^{\phi^{-1}} = z,$$

a contradiction as $z^\phi \ne z$. □

Lemma 6.3. *Take $z^\phi \in S - R$ for some $\phi \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$ so ϕ^{-1} extends to $C_S(z^\phi)$. Let $T = C_R(z^\phi)^{\phi^{-1}}$ and $R_0 = C_R(z^{\phi^{-1}})$. Then R_0 and T centralize each other, i.e. $[R_0, T] \le R_0 \cap T = 1$.*

Proof. Since $R \trianglelefteq S$, we have

$$[R_0, T] \le R. \tag{6.1}$$

Since $\langle z \rangle$ is fully $\mathcal{F}$-centralized we can extend ϕ to a map $\phi_1 \in \mathrm{Hom}_{\mathcal{F}}(C_S(z^{\phi^{-1}}), S)$ such that $z^{\phi^{-1}\phi_1} = z$ (we write ϕ_1 for the extension of ϕ to $C_S(z^{\phi^{-1}})$ to differentiate from the inverse of the extension of ϕ^{-1} to $C_S(z^\phi)$). Since $\phi^{-1}\phi_1 \in C_{\mathcal{F}}(\langle z \rangle)$ and R is strongly closed in $C_{\mathcal{F}}(\langle z \rangle)$ we get that

$$T^{\phi_1} = C_R(z^\phi)^{\phi^{-1}\phi_1} \le R \cap C_S(z^{\phi_1}) = C_R(z^{\phi_1}).$$

Since $R_0 \le C_S(z^{\phi^{-1}})$, $R_0^{\phi_1}$ is defined and so

$$[R_0, T]^{\phi_1} = [R_0^{\phi_1}, T^{\phi_1}] \le R \text{ as } T^{\phi_1} \le R. \tag{6.2}$$

Let $C^* = [R_0, T]$. By (6.1) and (6.2) we have that $C^* \le R$ and $(C^*)^{\phi_1} \le R$. Either $C^* = 1$ or it has the unique involution z of R. But if $z \in C^*$ then $z^{\phi_1} \in R$ by (6.2), a contradiction as $z^{\phi_1} \ne z$. Therefore $C^* = [R_0, T] = 1$ which completes the proof.

□

Before fixing T and R_0 as in the last chapter, we first show that if there is a distinct conjugate $z^{\phi^{-1}}$ of z such that $C_R(z^{\phi^{-1}}) > \langle z \rangle$ then whenever $|C_R(z^{\phi^{-1}})|$ is maximal then $C_R(z^{\phi^{-1}})$ is a maximal cyclic subgroup of R.

Lemma 6.4. *Take $z^{\phi^{-1}} \neq z$ for some $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$ such that $|C_R(z^{\phi^{-1}})|$ is maximal. If $|C_R(z^{\phi^{-1}})| \geq 4$ then $C_R(z^{\phi})$ and $C_R(z^{\phi^{-1}})$ are maximal cyclic subgroups of R (unique for $n > 3$).*

Proof. We extend ϕ to $C_S(z^{\phi^{-1}})$, and let $T = C_R(z^{\phi^{-1}})^{\phi}$, and $R_0 = C_R(z^{\phi})$. Observe that T contains z^{ϕ} as its unique involution and $|R_0| \leq |T|$ by the maximality of $|T|$. Let $N = N_{TR}(T)$. Suppose first that $N = TR$. In this case,

$$[T, R] \leq T \cap R = 1$$

which implies that $[z^{\phi}, R] = 1$, a contradiction by Lemma 6.2. This shows that $N \neq TR$. Since TR is a 2-group $N < N_{TR}(N)$, and so we can choose $s \in R$ such that

$$s \in N_{TR}(N) - N \text{ and } T^s \neq T. \tag{6.3}$$

We see T and T^s normalize each other as

$$N_{TR}(T) = N_{TR}(T)^s = N_{TR}(T^s).$$

It follows at once that $[T, T^s] \leq T \cap T^s$. If $z^{\phi} \in T \cap T^s$ then it follows that $(z^{\phi})^s = z^{\phi}$ and so $s \in C_R(z^{\phi}) = R_0$. By Lemma 6.3 we have that R_0, hence s, normalizes T contrary to (6.3). Thus $[T^s, T] \leq T \cap T^s = 1$, i.e. $TT^s = T \times T^s$.

Now notice that

$$T \times T^s \leq C_{TR}(z^{\phi}) = TR_0.$$

Since $|R_0| \leq |T|$ it follows that

$$T \times T^s = TR_0 \text{ and } T \cong R_0.$$

Next, we claim that $|TR : TR_0| = 2$. Since $R_0 \neq R$ we have that $|TR : TR_0| \geq 2$. Now let

$$W = \Omega_1(TR_0) = \langle\, z, z^\phi \,\rangle$$

and notice that

$$W \operatorname{char} TR_0 \trianglelefteq N_{TR}(TR_0) \leq N_{TR}(W).$$

Since $\mathrm{Aut}(W) \cong S_3$ and $C_{TR}(W) = C_{TR}(z^\phi) = TR_0$ we have that

$$1 \neq N_{TR}(W)/C_{TR}(W) = N_{TR}(W)/TR_0 \leq S_3.$$

It follows at once by order considerations that $|N_{TR}(W) : TR_0| = 2$. Consequently, all squares in $N_{TR}(W)$ are contained in TR_0. If $N_{TR}(W) \neq TR$ then it follows that there is some $u \in N_{TR}(N_{TR}(W)) - N_{TR}(W)$. Since $|T| \geq 4$ by assumption, z^ϕ is a square in T. Hence $(z^\phi)^u$ is a square in $N_{TR}(W)$ and we have that $z^{\phi u} \in TR_0$. Since $z^u = z$ we have that $W^u = W$, i.e. $u \in N_{TR}(W)$, a contradiction. Therefore it follows that

$$N_{TR}(W) = TR \text{ and so } |TR : TR_0| = 2$$

as claimed. It remains to show that R_0 is maximal cyclic.

Since R is generalized quaternion all its subgroups are cyclic or quaternion. But notice that $[T, R_0] = 1$ and $[T, T^s] = 1$. Since

$$T \times T^s = T \times R_0 = T^s \times R_0$$

we have that $T \leq Z(T \times T^s)$ which gives us that T is abelian and hence cyclic. Since $|R : R_0| = 2$ we that $R_0 \cong T$ is a maximal cyclic subgroup of R. □

We now fix notation and proceed similarly to the previous chapter. By Lemma 4.2 we have that $\langle z \rangle$ is not strongly closed so choose $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$ such that $|C_R(z^{\phi^{-1}})|$ is maximal. We extend ϕ to $C_S(z^{\phi^{-1}})$ and let $R_0 = C_R(z^{\phi})$ and let $T = C_R(z^{\phi^{-1}})^{\phi}$. By the previous lemma, either $T = \langle z^{\phi} \rangle$ or T is a maximal cyclic subgroup of R (which forces R_0 to be a cyclic as well).

As in the previous chapter, we notice that when $n = 3$, R_0 is S-invariant and when $n > 3$, R_0 is the unique maximal cyclic subgroup of R.

Next, we shall require a lemma assuring us that whenever two $\mathcal{F}$-conjugates of z commute, their product is $\mathcal{F}$-conjugate to z.

Lemma 6.5. *For any two distinct, commuting $\mathcal{F}$-conjugates z_1, z_2 of z, $z_1 z_2$ is also $\mathcal{F}$-conjugate to z.*

Proof. We may assume $z_1 = z$ and $z_2 = z^{\psi} \in S$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$. Let $X = R\langle z^{\psi} \rangle, U = \langle z, z^{\psi} \rangle$, and $C = C_X(z^{\psi}) = C_X(U)$. First, we observe that $R \not\leq C$ by Lemma 6.2 and hence $C < N_X(C)$. Now we claim U is the unique subgroup four-group in C. If there were another four-group $U_1 \neq U$ then $\langle U, U_1 \rangle$ would contain an E_8 which cannot happen in X. We have then that $U \,\mathrm{char}\, C$ and therefore $N_X(C)$ acts on U. It follows at once that for any $y \in N_X(C) - C$ we get $(z^{\psi})^y = z^{\psi y} = zz^{\psi}$ where $\psi c_y \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$. This completes the proof of the lemma. □

Lemma 6.6. *Suppose $C_R(z^{\psi}) = \langle z \rangle$ whenever $z^{\psi} \in S - \langle z \rangle$ for any $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$. Then $R\langle z^{\phi} \rangle$ is quasidihedral and strongly closed in S with respect to $\mathcal{F}$.*

Proof. First, we have that $P = R\langle z^{\phi} \rangle$ is quasidihedral by Suzuki's Lemma. Next we show that P is strongly involution closed. Since P is quasidihedral, it has only two P-classes of involutions: one consisting of just z (central element) and the other being $(z^{\phi})^P$. Thus if P is not strongly involution closed there is some

$$z^{\psi} \in S - P, \text{ for some } \psi \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S).$$

We claim that $C_P(z^{\psi}) = \langle z \rangle$. Suppose on the other hand that $|C_P(z^{\psi})| \geq 4$. We extend ψ^{-1} from $\langle z^{\psi} \rangle$ to $C_S(z^{\psi})$ so that

$$|C_P(z^{\psi})^{\psi^{-1}}| = |C_{P^{\psi^{-1}}}(z)| \geq 4.$$

Let $C = C_P(z^{\psi})$ and so $C^{\psi^{-1}}$ normalizes R and its S-invariant cyclic group of order 4, call it H. Therefore all squares in $C^{\psi^{-1}}$ centralize H so there is an involution $(z^{\beta})^{\psi^{-1}} \in C^{\psi^{-1}}$ for some $\beta \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, P)$ which is centralized by H. Thus $C_R((z^{\beta})^{\psi^{-1}}) > \langle z \rangle$ contrary to our hypothesis. Hence $C_P(z^{\phi}) = \langle z \rangle$. By Suzuki's Lemma we get that $P\langle z^{\psi} \rangle$ is quasidihedral, a contradiction as P is the quasisdihedral subgroup of index 2 in $P\langle z^{\psi} \rangle$ which cannot happen. Thus we conclude that P is strongly involution closed.

If P is not strongly closed then there is some $x \in P$ such that $x^{\psi} \in S - P$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, S)$. By the arguments above x is not an involution and we also know that some power of x is z. Since P is strongly involution closed, $z^{\psi} \in P$. First suppose that $z^{\psi} \neq z$. Then as above, z^{ψ} centralizes a cyclic group of order 4 in R since x^{ψ} normalizes R. So $|C_R(z^{\psi})| \geq 4$ which again contradicts our hypothesis. So $z^{\psi} = z$, i.e. $\psi \in C_{\mathcal{F}}(\langle z \rangle)$.

Since P is quasidihedral we have that its three maximal subgroups (of index 2) are M_1 (cyclic), M_2 (dihedral), and R (generalized quaternion). This gives us that $\Phi(P)$ is the maximal cyclic subgroup of R and we have that $|P : \Phi(P)| = 4$ by Lemma 3.7.

Then it follows that

$$P/\Phi(P) = M_2/\Phi(P) \times R/\Phi(P) = \langle\, \overline{z}_1 \,\rangle \times \langle\, \overline{z}_2 \,\rangle$$

where we can choose z_1 to be an involution in M_2. Because by our overall hypotheses R is strongly closed in $C_{\mathcal{F}}(\langle\, z \,\rangle)$ we have that $x \notin R$. If $x \in M_2$ then since x is not an involution, we have that $x \in \Phi(P) \leq R$ which again, cannot be the case. Thus we have that $x \in M_1$. Since $|R : \Phi(P)| = 2$ and $x \notin R$ we have further that $M_1 = \langle\, x \,\rangle$. Since

$$\langle\, x^2 \,\rangle = \Phi(P) \leq R$$

we have that $(x^{\psi})^2 \in R$ (again by strong closure) and so

$$\Phi(P) = \langle\, (x^{\psi})^2 \,\rangle \text{ i.e., } \psi \in N_{\mathcal{F}}(\Phi(P)) \leq C_{\mathcal{F}}(\langle\, z \,\rangle).$$

Since R is strongly closed (with respect to $N_{\mathcal{F}}(\Phi(P))$) we have that $R/\Phi(P)$ is strongly closed in $S/\Phi(P)$ with respect to $N_{\mathcal{F}}(\Phi(P))/\Phi(P)$. If $\overline{z}_1^{\overline{\psi}} \in \overline{S}$ for $\psi \in N_{\mathcal{F}}(\Phi(P))$ then $z_1^{\psi} \in P$ as P is strongly involution closed. Since $z_1^{\psi} \neq z$ we get $z_1^{\psi} \in M_2$. So $\langle\, \overline{z_1} \,\rangle$ is strongly closed and hence $\overline{P} = \langle\, \overline{z_1} \,\rangle \times \langle\, \overline{z_2} \,\rangle$ is strongly closed in $S/\Phi(P)$ with respect to $N_{\mathcal{F}}(\Phi(P))/\Phi(P)$ by Lemma 4.1. We therefore conclude that P is strongly closed in S with respect to $N_{\mathcal{F}}(\Phi(P))$, contrary to the existence of $x^{\psi} \in S - P, \psi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, x \,\rangle, S)$. This completes the proof of the lemma. □

To complete the proof of Theorem 6.1 we now consider when $R_0 = C_R(z^{\phi})$ is maximal in R_0 by Lemma 6.4. The next lemma is merely a reproduction of Lemma 5.7 with z^{ϕ} in place of z^g.

Lemma 6.7. *$P = TR \cong Z_{2^{n-1}} \wr Z_2$.*

Proof. This is Lemma 5.7. □

We maintain the notation of $P = TR = TR_0\langle t \rangle = (\langle a \rangle \times \langle b \rangle)\langle t \rangle$ where $s \in R - R_0$, $t = z^{\phi}s$ is a wreathing involution, $T = \langle a \rangle$, and $T^s = \langle b \rangle$.

We move on to our next task: showing that $W = \langle z, z^{\phi} \rangle \trianglelefteq S$.

Lemma 6.8. *$W = \langle z, z^{\phi} \rangle$ is normal in S, and so $|S : C_S(z^{\phi})| = 2$.*

Proof. Let $A = TR_0$ so if W is not normal in S then neither is A as $W \operatorname{char} A$. Suppose this is the case and take $u \in N_S(N_S(A)) - N_S(A)$ so

$$A^u \leq N_S(A) = N_S(A^u) = N_S(A)^u.$$

We first suppose that $W^u = W$. This forces $z^{\phi} \in W^u$ so that $(z^{\phi})^u = z^{\phi}$ or $(z^{\phi})^u = zz^{\phi}$. Since $TR = A\langle t \rangle \cong Z_{2^{n-1}} \wr Z_2$ we can replace u by ut if necessary to get $(z^{\phi})^u = z^{\phi}$ so assume $u \in C_S(z^{\phi})$. We now extend ϕ^{-1} to a map $\phi_1^{-1} \in \operatorname{Hom}_{\mathcal{F}}(C_S(z^{\phi}), S)$ such that $z^{\phi\phi_1^{-1}} = z$ (we write ϕ_1^{-1} for the extension of ϕ^{-1} to $C_S(z^{\phi})$ to differentiate from the inverse of the extension of ϕ to $C_S(z^{\phi^{-1}})$ which we fixed following Lemma 6.4). Since $\phi\phi_1^{-1} \in C_{\mathcal{F}}(\langle z \rangle)$ and R is strongly closed in $C_{\mathcal{F}}(\langle z \rangle)$ we get that

$$T^{\phi_1^{-1}} = C_R(z^{\phi^{-1}})^{\phi\phi_1^{-1}} \leq R \cap C_S(z^{\phi_1^{-1}}) = C_R(z^{\phi_1^{-1}}). \tag{6.4}$$

Since $C_R(z^{\phi^{-1}})$ is of maximal order, Lemma 6.4 and (6.4) give us that $T^{\phi_1^{-1}} = C_R(z^{\phi_1^{-1}})$. It follows by the comments following Lemma 6.4 that $u^{\phi_1^{-1}}$ normalizes $T^{\phi_1^{-1}}$. From this we get that u normalizes T and since u normalizes R_0 as well, it follows that $u \in N_S(A)$, a contradiction. This shows that $W^u \neq W$.

Next, notice that $[A, A^u] \leq A \cap A^u = R_0$ and also

$$T \leq N_S(A) = N_S(A^u) \leq N_S(W^u).$$

This forces $[(z^{\phi})^u, T] \le W^u$ and hence

$$[(z^{\phi})^u, T] \le R_0 \cap W^u = \langle\, z \,\rangle.$$

Now let $X = \langle\, (z^{\phi})^u, z, T \,\rangle$ so that

$$\overline{X} = X/\langle\, z \,\rangle \cong Z_2 \times Z_{2^{n-1}}.$$

By Lemma 3.6 we have that $\overline{X}$ cannot act faithfully on R. Thus at least one of $\overline{z^{\phi}}$, $\overline{(z^{\phi})^u}$, or $\overline{z^{\phi}(z^{\phi})^u}$ centralizes R. Since z^{ϕ} is a square in T we have that z^{ϕ} centralizes W^u and in particular, $[z^{\phi}, (z^{\phi})^u] = 1$. By Lemma 6.6, z is $\mathcal{F}$-conjugate to $z^{\phi}(z^{\phi})^u$ so no matter which involution in $\overline{X}$ centralizes R we have a contradiction as $z^{\mathcal{F}} \cap C_S(R) = \{z\}$ by Lemma 6.2. Since S centralizes z it follows by order considerations that $|S : C_S(W)| = |S : C_S(z^{\phi})| = 2$. □

We remark that as $|S : C_S(z^{\phi})| = 2$ we can view ϕ as an isomorphism from $C_S(z^{\phi^{-1}})$ into $C_S(z^{\phi})$. Now we determine the structure of $C_S(z^{\phi})$.

Lemma 6.9. *Let $A = TR_0 \cong Z_{2^{n-1}} \times Z_{2^{n-1}}$. Then there is some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(z^{\phi}))$ of 3-power order such that*

$$C_S(z^{\phi}) = A \rtimes C \text{ where } C = C_{C_S(z^{\phi})}(\langle\, \alpha \,\rangle) \text{ and } A \trianglelefteq S.$$

Proof. First we find the required $\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(z^{\phi}))$. For $s \in R - R_0$ we get that $z^{\phi c_s} = zz^{\phi}$ and $z^{c_s} = z$ so c_s is an involution in $\mathrm{Aut}_S(W)$. By Lemma 6.6 there is some $s' \in R$ such that $z^{\phi^{-1}c_{s'}} = zz^{\phi^{-1}}$. So $c_{s'}^* = \phi^{-1}c_{s'}\phi \in \mathrm{Aut}_{\mathcal{F}}(C_S(z^{\phi}))$ is another involution in $\mathrm{Aut}_{\mathcal{F}}(W)$. Thus $\alpha = c_s c_{s'}^*$ has order 3 as an automorphism of W, and order divisible by 3 as an automorphism of $C_S(z^{\phi})$. We assume that α is of 3-power order after taking an appropriate power. Since α has a $2'$ action on $C_S(z^\phi)$, Fitting's Lemma (Lemma 3.11) gives us that

$$C_S(z^\phi) = [C_S(z^\phi), \alpha]C, \text{ where } C = C_{C_S(z^\phi)}(\langle \alpha \rangle).$$

First, we show that $[C_S(z^\phi), \alpha] \leq A$. Since $s \in R$ we know

$$[C_S(z^\phi), s] \leq C_S(z^\phi) \cap R = R_0.$$

Now, take $g \in C_S(z^\phi)$ so then

$$\begin{aligned} [g, c^*_{s'}] = g^{-1}g^{c^*_{s'}} &= g^{-1}g^{\phi^{-1}c_{s'}\phi} \\ &= ((g^{-1})^{\phi^{-1}}g^{\phi^{-1}c_{s'}})^\phi \\ &= [g^{\phi^{-1}}, c_{s'}]^\phi \in (C_S(z^{\phi^{-1}}) \cap R)^\phi = T. \end{aligned}$$

It follows then from the above that $[C_S(z^\phi), c^*_{s'}] \leq T$. Observe $c^*_{s'} \in C_{\mathcal{F}}(z)$ so we get $R_0^{c^*_{s'}} = R_0$ by our overall hypothesis. Since

$$[g, \alpha] = [g, c_s c^*_{s'}] = [g, c^*_{s'}][g, c_s]^{c^*_{s'}} \in TR_0^{c^*_{s'}} = A \qquad \text{for all } g \in C_S(z^\phi)$$

we conclude that $[C_S(z^\phi), \alpha] \leq TR_0 = A$. Since α acts nontrivially on the four-group $W = \Omega_1(A)$, $C \cap A = 1$, and $[C_S(z^\phi), \alpha] = A$. Thus $C_S(z^\phi) = A \rtimes C$ as desired.

Finally, we observe that we have shown that

$$S = C_S(z^\phi)\langle s \rangle = (A \rtimes C_{C_S(z^\phi)}(\langle \alpha \rangle))\langle s \rangle \text{ with } A = [C_S(z^\phi), \alpha].$$

From this it follows that $A \trianglelefteq S$ since c_s inverts α. This completes the proof. $\square$

We maintain the notation of Lemma 6.9 above for the remainder of the chapter. We point out for future reference that since c_s inverts α it follows that s normalizes C. But since R is normal in S we also get that $[s, C] \leq R \cap C = 1$.

We record another important consequence of the preceding lemma.

Lemma 6.10. *C acts faithfully on A, and so $C_S(A) = A$. Moreover, C induces the same action on all maximal cyclic subgroups of A and so C is isomorphic to a subgroup of*

$$(\mathbb{Z}/2^{n-1}\mathbb{Z})^{\times} \cong \mathrm{Aut}(Z_{2^{n-1}}) \cong Z_2 \times Z_{2^{n-3}}.$$

Proof. Let $C^* = C_S(A) \cap C \trianglelefteq S$. If $C^* \neq 1$ then take an involution $u \in Z(S) \cap C^*$. By our overall hypotheses, R is strongly closed in $C_{\mathcal{F}}(\langle u \rangle)$. Since $u \in C$, $u^{\alpha} = u$; and since $\alpha \in C_{\mathcal{F}}(\langle u \rangle)$ but $\langle z, z^{\alpha} \rangle$ is a 2-group, $z^{\alpha} = z$ by strong closure, a contradiction as $z^{\alpha} = z^{\phi}$.

The last assertion is just Lemma 3.15. □

Next, we will show in the case $R \cong Q_8$ that S is completely determined.

Theorem 6.11. *If $R \cong Q_8$ then $S = TR \cong Z_4 \wr Z_2$ is wreathed or $S \cong (Q_8 * Q_8)\langle t \rangle$ is of $G_2(q)$-type. Furthermore, $\mathcal{F}$ is of type $L_3(q)$ or $G_2(q)$ for suitable odd q.*

Proof. The determination of S completely follows Theorem 5.11. The second statement follows from [Oli11]. □

Now we consider the case when $n \geq 4$.

Theorem 6.12. *If $n \geq 4$ then $\mathcal{F}$ is the fusion system of $L_3(q)$ or $G_2(q)$ for suitable odd q.*

Proof. By the preceding lemmas we've established that $S = A(C\langle t \rangle)$ where $C \leq Z_2 \times Z_{2^{n-3}}$. If $C = 1$ then $S \cong Z_{2^{n-1}} \wr Z_2$ and so by [Oli11] $\mathcal{F}$ is the fusion system of

$L_3(q)$ for suitable odd q. Suppose now that $C \neq 1$. We wish to show that otherwise $C = \langle\, r \,\rangle$ where r acts by inversion on A thereby forcing S to be of $G_2(q)$-type. Since W is not strongly closed there is some $z^\psi \in S - A$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, z \,\rangle, S)$.

First, we check that if $z^\psi \in C$ then z^ψ does not act as a modular automorphism on A. If so, then $C_R(z^\psi) \cong Q_{2^{n-1}}$ by Lemma 3.14. As $|C_R(z^\psi)|$ is maximal we have a contradiction by Lemma 6.4; thus no conjugate of z acts as the modular automorphism of R_0.

Now we wish to show that $C = \langle\, r \,\rangle$ where $r = 1$ or r is a non-modular involution acting on A. Suppose not so there exists a modular involution d in C that centralizes the subgroup of index 2 in R_0. Moreover, $C = \langle\, i \,\rangle \times \langle\, e \,\rangle$ where either $i = 1$ or i acts by inversion on A, and where d is the unique involution in $\langle\, e \,\rangle$. Then in this case

$$C_S(d) = (\langle\, a^2 \,\rangle \times \langle\, b^2 \,\rangle)(C\langle\, t \,\rangle)$$

which is of index 4 in S. Let $M = A\langle\, i \,\rangle\langle\, t \,\rangle$ so $S/M \cong Z_{2^m}$. By the Thompson Transfer Lemma for fusion systems (Lemma 4.13) there exists a $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, d \,\rangle, M)$ and ψ can be chosen such that $\langle\, d^\psi \,\rangle$ is fully centralized.

If $d^\psi \in W$ we have a contradiction by the observation above that any $\mathcal{F}$-conjugates of z in C must convert the maximal cyclic subgroups of A into dihedral or quasidihedral groups. Thus $d^\psi \notin A$. We must have then that $\overline{d^\psi} \in \langle\, \bar{i} \,\rangle \times \langle\, \bar{t} \,\rangle$ in $\overline{S} = S/A$. If $\overline{d^\psi} = \bar{i}$ then

$$C_S(d^\psi) \leq W(C\langle\, t \,\rangle)$$

which is of index $2^{n-2}2^{n-2} > 4$ in S. If $\overline{d^\psi} = \bar{t}$ then

$$C_S(d^\psi) \leq \langle\, ab \,\rangle(C\langle\, t \,\rangle)$$

which is of index $2^{n-1} > 4$ in S. Finally, if $\overline{d^\psi} = \overline{it}$ then

$$C_S(d^\psi) \leq \langle\, ab^{-1} \,\rangle (C\langle\, t \,\rangle)$$

which is of index $2^{n-1} > 4$ in S. Whatever the case, we get $|C_S(d)| > |C_S(d^\psi)|$ which contradicts the fact that $\langle\, d^\psi \,\rangle$ is fully centralized. Thus $C = \langle\, r \,\rangle$ where r is a non-modular involution and

$$S = A\langle\, r, t \,\rangle.$$

If r inverts A then by Lemma 3.9, $S \cong (Q_{2^n} * Q_{2^n})\langle\, t \,\rangle$ and by [Oli11] it follows that $\mathcal{F}$ is the fusion system of $G_2(q)$ for suitable odd q.

It remains to consider the case in which r has the quasidihedral action on A. Since W is not strongly closed, we have $z^\psi \in S - A$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, z \,\rangle, S)$. Now because r has the quasidihedral action on R_0 and t inverts R_0, rt acts modularly on R_0. Therefore z^ψ must equal r or t (mod A) by the argument in the second paragraph of this proof. We apply Thompson's Transfer Lemma again to S/M where $M = A\langle\, r \,\rangle$ or $M = A\langle\, t \,\rangle$ to get that all involutions in S are $\mathcal{F}$-conjugate to z. But then rt is $\mathcal{F}$-conjugate to z which again gives a contradiction by the second paragraph of this proof. This completes the proof of Theorem 1. $\square$

Chapter 7

The Group-Theoretic Proof for $L_2(q)$

Let G be a finite simple group and t an involution such that $C_G(t)$ has a standard component L of type $L_2(q)$ with $q > 9$ for q odd. We fix the notation $N = N_G(L)$ and $C = C_G(L)$. Also, let $S \in Syl_2(N)$, and let $S^* \in Syl_2(G)$ with $S \leq S^*$. Let $Q = S \cap C \in Syl_2(C)$. Recall that a component is standard if

(i) C is tightly embedded in G with $N = N_G(C)$.

(ii) $[L, L^g] \neq 1$, for all $g \in G$.

When $q > 9$, by Lemma 3.19 $L \cong L_2(q)$ has the important property that for any four-group U acting on L,

$$L \leq \Gamma_{U,1}(L) = \langle\, C_L(w) \mid w \in U^\sharp \,\rangle.$$

We treat the situation where Q has 2-rank greater than 1. So in this chapter we characterize simple groups with an involution centralizer containing a standard component of type $L_2(q)$, q odd and $q > 9$. That is, our aim is to prove the following theorem.

Theorem 7.1. *When $m(Q) > 2$, S^* contains a proper, nontrivial strongly closed subgroup. In particular, one of the following holds:*

(i) Q is strongly closed, or

(ii) Q is a dihedral subgroup of index 2 contained in a strongly closed dihedral subgroup of S^*.

While this result is established in [Foo78] we provide a strictly "fusion-theoretic proof" which gives us direction in proving a fusion systems version in the next chapter.

For the remainder of this chapter over all $g \in G - N$ choose $T \in Syl_2(C^g \cap N)$ to be of maximal rank and, subject to this, to be of maximal order. If necessary, we can also adjust g by an element of N such $T \leq S$. Observe that Q is strongly closed in S with respect to N: if $h \in N$ and $q^h \in S$ for some $q \in Q$ then $q^h \in S \cap C = Q$ (because Q is a Sylow 2-subgroup of the normal subgroup C of N).

Lemma 7.2. *If $T = 1$ then Q is strongly closed in S^* with respect to G.*

Proof. If $T = 1$ but Q is not strongly closed then Q is not strongly closed in $N_G(Q)$. By the observation above, there is some $q \in Q$ and $g \in G - N$ such that $q^g \in N_G(Q) - Q$. Thus $C^g \cap N_G(Q) \leq C^g \cap N$ has even order, contrary to $T = 1$. □

Lemma 7.3. *T is non-trivial, $m(T) = 1$, and T is either cyclic or generalized quaternion.*

Proof. If $m(T) = 1$ then by [Gor80, 5.4.10] T is either cyclic or generalized quaternion. Suppose now $m(T) > 1$. Then there is a four-group $W \leq T$ which acts on L and so by hypothesis, $\Gamma_{W,1}(L) = L$. By tight embedding $C_L(w)$ normalizes L^g for each $w \in W^\sharp$ so $L \leq N^g$. Now we seek a four-group in $Q \cap N^g$ so that (by the above arguments) $L^g \leq N$. If this is true then L and L^g are both components of $N \cap N^g$. But then $[L, L^g] = 1$ by [Asc00, 31.5], i.e., distinct components commute ($g \notin N$ so $L \neq L^g$). This would contradict the fact that L is a standard component.

If Q normalizes T then $[T, Q] \leq T \cap Q = 1$. This means there is a four-group in Q centralizing T and hence acting on C^g by tight embedding. As this would complete the proof by the above remarks, we may assume Q does not normalize T. Thus $N_{TQ}(T) < TQ$. By the basic theory of p-groups there is some element $s \in Q$ that normalizes $N_{TQ}(T)$ but does not normalize T. It follows that T and T^s normalize each other as

$$N_{TQ}(T) = N_{TQ}(T)^s = N_{TQ}(T^s).$$

So $[T, T^s] \leq T \cap T^s$. If $T \cap T^s \neq 1$ then $s \in N^g$ by tight embedding so then $\langle T, T^s \rangle \leq C^g \cap N$. This forces $T^s = T$ as $T \in Syl_2(C^g \cap N)$. So $T \cap T^s = 1$ and we have that $[T, T^s] = 1$. By assumption $m(T) > 1$ and so there is four-group U with

$$U \leq [s, T] \leq Q \cap N_{TQ}(T).$$

Since U centralizes T, tight embedding gives us that $U \leq N^g$, thereby providing the necessary four-group and completing the proof. □

From this we obtain the following corollaries.

Corollary 7.4. *Q is weakly closed in S with respect to G and S is a Sylow 2-subgroup of G (i.e. $S = S^*$).*

Proof. Suppose $Q^h \leq S$ for some $h \in G$. If $h \in N$ then

$$Q^h \leq S \cap C = Q.$$

Otherwise, $h \notin N$ and so $Q^h \leq C^h \cap N$ contains a four-group which contradicts the maximality of the rank of T. Thus Q is weakly closed in N, hence also in S^* by Lemma 3.3. □

7.1 The $|T|$=2 case

In this section we assume $|T| = 2$. First we show that TQ and Q are both dihedral. We use repeatedly the observation that whenever $w^g \in S - Q$ for some $w \in Q$

and $g \in G$, then $|C_Q(w^g)| = 2$ – otherwise the maximality of $|T|$ is violated since $C_Q(w^g) \leq N^g$ by tight embedding (after replacing g by g^{-1}).

Lemma 7.5. *If $|T| = 2$ then Q is dihedral, TQ is dihedral and, in particular, $m(Q)$=2.*

Proof. Let $T = \langle\, t\,\rangle$ and consider $Q\langle\, t\,\rangle$. If $|C_Q(t)| \geq 4$ then $C_Q(t) \leq N^g$ by tight embedding. So $C_Q(t) \leq N^g \cap C$ which contradicts the maximality of T (after interchanging g and g^{-1}). Thus $|C_Q(t)| = 2$ and so by Suzuki's Lemma TQ is dihedral or quasidihedral. Since Q is a maximal subgroup of TQ and $m(Q) \geq 2$, this forces both Q and TQ to be dihedral. □

We now show in the situation of Lemma 7.5 that S contains a strongly closed dihedral subgroup.

Theorem 7.6. *If $|T| = 2$ then $P = TQ$ is a strongly closed dihedral group.*

Proof. Because $|T| = 2$, Q is not strongly involution closed. Take an involution $w \in Q$ such that $T = \langle\, w^g\,\rangle \not\leq Q$. By Lemma 7.5, $P = Q\langle\, w^g\,\rangle$ must be dihedral so let $Z(P) = \langle\, z\,\rangle$. This forces all non-central involutions in Q to be conjugate. The Z^*-theorem ([Gla66]) gives us that z is not isolated. Hence z conjugates outside Q (in which case we could replace w^g by, say, z^g) or z conjugates into $Q - \langle\, z\,\rangle$. In the latter case, the above observation would force w and z to be conjugate so again we could replace w^g by z^g. So in all cases we may choose g so that $P = Q\langle\, z^g\,\rangle$, and in all cases every involution of P is G-conjugate to one of Q.

We aim to show first that P is strongly involution closed. Assume not and by the preceding paragraph take $w^h \in S - P$ for some involution $w \in Q$ and $h \in G$. If $P\langle\, w^h\,\rangle$ is dihedral then we arrive at a contradiction as $Q \trianglelefteq P\langle\, w^h\,\rangle$ but a dihedral group does not have a normal dihedral subgroup of index 4. By Suzuki's Lemma this forces $|C_P(w^h)| > 2$. We observe that if $C_P(w^h)$ is cyclic then an element of order 4 or greater in Q centralizes w^h. By tight embedding, this contradicts the maximality of T. So $C_P(w^h)$ contains a four-group (not contained in Q) which in turn must contain, up to conjugation, z^g.

Assume that $|Q| > 4$. Now w^h and z^g both invert the maximal cyclic subgroup of Q and hence $z^g w^h$ centralizes that maximal cyclic subgroup of Q. If we can show that w is conjugate to $z^g w^h$ we get that the maximal cyclic subgroup of Q would centralize $z^g w^h$, i.e a conjugate of z would centralize this maximal cyclic subgroup. This would contradict the maximality of T. Since w^h centralizes z^g, we may assume that $w^h \in S^g$ (possibly after an adjustment via Sylow's Theorem). We've established that $Q^g\langle\, w^h \,\rangle$ must be dihedral with center $\langle\, z^g \,\rangle$, so w^h is conjugate to $z^g w^h$ in this dihedral group. Thus P is strongly involution closed when $|Q| > 4$.

Assume, on the other hand, that Q is a four-group. Let $Q = \langle\, u, z \,\rangle$. It follows then that w^h and z^g both conjugate u into zu. Hence $z^g w^h$ centralizes all of Q. After any necessary Sylow-adjustments, we may assume that $w^h \in S^g$. Then $Q^g\langle\, w^h \,\rangle$ is dihedral with center $\langle\, z^g \,\rangle$ and hence w^h is conjugate to $z^g w^h$. We have found a conjugate of z outside of Q that centralizes Q which contradicts the maximality of T. Thus P is strongly involution closed in this case too.

Now we show that P is strongly closed (no matter the order of Q). Take $y \in P$ of order 4 or more such that $y^h \in S - P$ for some $h \in G$. By tight embedding and our hypothesis on T we have that $y \notin Q$. Since the only elements of order 4 or greater are contained in the maximal cyclic subgroup of P and $|Q : \Phi(P)| = 2$, this forces $\langle\, y \,\rangle$ to be the maximal cyclic subgroup in P. Thus $\langle\, y^2 \,\rangle = \Phi(P)$ and

$$\langle\, (y^2)^h \,\rangle = \langle\, y^2 \,\rangle = \Phi(P), \text{ that is, } h \in N_G(\Phi(P)) \leq N.$$

Now let

$$N_G(\Phi(P))/\Phi(P) = \overline{N_G(\Phi(P))}.$$

Note that $\overline{P} = \langle \overline{Q} \rangle \times \langle \overline{z^g} \rangle$. We will show that $\overline{P}$ is strongly closed in $\overline{S}$ with respect to $\overline{N_G(\Phi(P))}$, which is sufficient to complete the proof as $h \in N_G(\Phi(P))$. Notice that the cosets $\overline{Q}$ and $\overline{z^g}$ contain involutions whereas $\overline{Qz^g} = \overline{y}$ does not. Since Q is strongly closed in N, $\overline{Q}$ is strongly closed in $\overline{N_G(\Phi(P))}$. These observations plus the fact that P is strongly involution closed forces $\langle \overline{z^g} \rangle$ to be strongly closed in $\overline{N_G(\Phi(P))}$ as well. By Lemma 3.12 we conclude that $\overline{P}$ is strongly closed in $\overline{N_G(\Phi(P))}$ and hence P is strongly closed as well, a contradiction. □

We've produced a strongly closed subgroup when $|T| = 2$ so we henceforth assume that $|T| > 2$.

7.2 The $|T| > 2$ case

Note in this case $|Q| \geq 8$ because T is a proper subgroup of Q^g.

Lemma 7.7. *Let $x \in T$ be an element of order 4 and set $P = \langle x \rangle Q$. Then*

(i) Q does not contain a four-group normalized by P,

(ii) $|Q : C_Q(x^2)| = 2$, and

(iii) $m(Q) = 2$.

Proof. We first show that P contains a normal four-group. Suppose the contrary so that P is a dihedral or quasidihedral group by [Gor80, 5.4.10]. In either case P contains the normal subgroup Q of index 4. This is contrary to the fact that the cyclic group $\Phi(P)$ is the unique normal subgroup of index 4 in any dihedral or quasidihedral group of order at least 2^4. Thus there exists U a normal four-group in P.

Note that x^2 centralizes U hence $U \leq C_G(x^2) \leq N^g$ by tight embedding. If $U \leq Q$, Lemma 7.3 and the maximality of $m(T)$ give a contradiction. Thus there exists a four-group $U \trianglelefteq P$ and U cannot be chosen to lie in Q. This forces $Z(Q)$ to be cyclic. Let $\Omega_1(Z(Q)) = \langle z \rangle$ so that $z \in U$. It follows at once that $U = \langle x^2q, z \rangle$ for some $q \in Q$. Since x^2 centralizes U, $[x^2, q] = 1$ and $q^2 = 1$. Since x^2 centralizes $\langle q, z \rangle$ but, as above, does not centralize *any* four-group in Q, $q = 1$ or $q = z$; and in either case,

$$U = \langle x^2, z \rangle.$$

Since $U \trianglelefteq P$ we have $|P : C_P(x^2)| \leq 2$. But if $[Q, x^2] = 1$ then $Q \leq N^g$ contrary to the maximality of the rank of T. Thus $|Q : C_Q(x^2)| = 2$.

If $m(Q) \geq 3$ then Q has a subgroup $E \cong E_8$ and so $E \cap C_P(x^2)$ contains a four-group. Again by tight embedding, $E \cap C_P(x^2) \leq N^g$ which contradicts the maximality of the rank of T. Thus $m(Q) = 2$.

□

For the remainder of this section, preserve the notation from the previous lemma and all lemmas that follow. We now show that Q is dihedral or quasidihedral.

Lemma 7.8. *Q^g is a dihedral or quasidihedral group with maximal cyclic subgroup T, and $N_{TQ}(T) = T \times T^s$ for some $s \in Q - N_{TQ}(T)$ with $s^2 \in N_{TQ}(T)$.*

Proof. We first show that T is abelian and hence cyclic by Lemma 7.3. If Q normalizes T then by tight embedding $Q \leq N^g$ which contradicts the maximality of T. Thus there exists some $s \in Q$ such that

$$s \in N_{TQ}(N_{TQ}(T)) - N_{TQ}(T), \text{ so that } T^s \neq T.$$

Clearly T and T^s normalize each other so that $[T, T^s] \leq T \cap T^s$. If $T \cap T^s \neq 1$ then tight embedding forces $s \in N^g$ so that $T^s \leq C^g \cap N$. Since $T \in Syl_2(C^g \cap N)$ this forces $T^s = T$, a contradiction. Hence we conclude that $TT^s = T \times T^s$. By the previous lemma, $|Q : C_Q(x^2)| = 2$ which by maximality forces T to be of index 2 in Q^g. It follows that

$$|TQ : T \times T^s| = 2.$$

Let $Q_0 = Q \cap (T \times T^s)$ so that because $[T, s] \leq Q_0$ we have

$$T \times T^s = T \times Q_0 = T^s \times Q_0.$$

Since $[Q_0, T] = 1$ and $[T^s, T] = 1$ it follows that $T \leq Z(T \times T^s)$. Therefore T is abelian and hence cyclic.

By [Gor80, 5.4.4] Q must be a modular, dihedral, or quasidihedral group. If Q is modular, then $\Omega_1(Q)$ is a characteristic four-group; but we established in the previous lemma Q may not posses such a subgroup. Thus we conclude that Q is dihedral or quasidihedral. □

We now show that TQ is a wreathed product and consequently, that Q is dihedral.

Lemma 7.9. *TQ is a wreathed product.*

Proof. By the previous lemma $TQ_0 = T \times T^s = \langle\, a \,\rangle \times \langle\, b \,\rangle$ where $T = \langle\, a \,\rangle$ and $b = a^s$ for some $s \in Q - Q_0$ with $s^2 \in Q_0$. Since $s \in Q$, $s^2 = 1$ or $s^2 = z$; and because $z \in Z(S)$ we get in either case

$$b^s = (a^s)^s = a^{s^2} = a.$$

So s interchanges a and b. Regardless of whether Q is dihedral or quasidihedral (both have non-central involutions) we may assume $|s| = 2$. Then s is a wreathing involution and we have

$$TQ_0\langle s \rangle \cong Z_{2^{n-1}} \wr Z_2.$$

□

Corollary 7.10. *Q is a dihedral group.*

Proof. If $|Q| = 8$ then Q is dihedral, so suppose $n > 3$. Since $Q \trianglelefteq TQ \cong Z_{2^{n-1}} \wr Z_2$, Q cannot be quasidihedral by Lemma 3.8. □

Henceforth let $\langle z \rangle = Z(Q)$. Next we prove several useful properties of conjugates in z^G.

Lemma 7.11. $z^G \cap C_S(Q) = \{z\}$.

Proof. Since $C_G(z^h) \leq N^h$ for all $h \in G$, this is immediate from Lemma 7.3 and the maximality of $m(T)$. □

Lemma 7.12. *For any two distinct commuting conjugates z_1, z_2 of z, z_1z_2 is also conjugate in G to z.*

Proof. We may assume $z_1 = z$ and $z_2 = z^h \in S$. Let $X = Q\langle z^h \rangle$, $U = \langle z, z^h \rangle$, and $C^* = C_X(z^h) = C_X(U)$. First, we observe that $Q \not\leq C^*$ by Lemma 7.11 and hence $C^* < N_X(C^*)$. Now we claim U is the unique four-group in C^*. If there were another four-group $U_1 \neq U$ then $\langle U, U_1 \rangle$ would contain a subgroup $E \cong E_8$. Then $E \cap Q$ has rank at least two and centralizes z^h which contradicts the fact that $m(T) = 1$. Then we have that $U \operatorname{char} C^*$ and therefore $N_X(C^*)$ acts on U. It follows at once that for any $y \in N_X(C^*) - C^*$ we get $(z^h)^y = z^{hy} = zz^h$ as desired. □

The next step is to mimic Lemma 5.8 to establish that $C_S(z^g)$ is of index 2 in S.

Lemma 7.13. $W = \langle z, z^g \rangle$ *is normal in S, and so* $|S : C_S(z^g)| = 2$.

Proof. Let $A = TQ_0$ so if W isn't normal in S then neither is A, as W char A. Suppose this is the case and take $u \in N_S(N_S(A)) - N_S(A)$ so

$$A^u \leq N_S(A) = N_S(A^u) = N_S(A)^u.$$

We first suppose that $W^u = W$. This forces $z^g \in W^u$ so that $z^{gu} = z^g$ or $z^{gu} = zz^g$. Since $TQ = A\langle\, s \,\rangle \cong Z_{2^{n-1}} \wr Z_2$ we replace u by us if necessary (which still does not normalize A) to get $z^{gu} = z^g$ so $u \in C_G(z^g)$. Since $\langle\, T, Q_0, u \,\rangle \leq S$ is a 2-group contained in $C_G(z^g)$ we can choose $k \in C_G(z^g)$ such that

$$\langle\, T, Q_0, u \,\rangle^k = \langle\, T^k, Q_0^k, u^k \,\rangle \leq S^g.$$

But Q^g is strongly closed in $C_G(z^g) \leq N^g$ which forces $T^k = T$. Since u normalizes Q_0 and u^k normalizes T it follows then that u^k normalizes $T \times Q_0^k$. Hence u normalizes $A = T \times Q_0$ contrary to assumption. This shows that $W^u \neq W$.

By the observation above that A and A^u normalize each other we get $[A, A^u] \leq A \cap A^u$. But $Q_0 \trianglelefteq S$, so $Q_0 \leq A \cap A^u$. Since $W^u \neq W$ it follows that z is the unique involution in $A \cap A^u$. Thus $A \cap A^u = Q_0$ and we get that $[A, A^u] \leq Q_0$. But then

$$T \leq N_S(A) = N_S(A^u) \leq N_S(W^u).$$

This forces $[z^{gu}, T] \leq W^u$ and hence

$$[z^{gu}, T] \leq Q_0 \cap W^u = \langle\, z \,\rangle.$$

Now let $X = \langle z^{gu}, z, T \rangle$ so that

$$\overline{X} = X/\langle z \rangle \cong Z_2 \times Z_{2^{n-1}}.$$

By Lemma 3.6 we have that $\overline{X}$ can't act faithfully on Q. Thus at least one of $\overline{z^g}, \overline{z^{gu}}$, or $\overline{z^g z^{gu}}$ centralizes Q. We see from Lemma 7.11 that $z^G \cap C_G(Q) = \{z\}$ hence $z^g, z^{gu} \notin C_S(Q)$. Since z^g is a square in T which acts on W^u we have that $[z^g, z^{gu}] = 1$ and so by Lemma 7.12 we have that $z^g z^{gu}$ is conjugate to z; this contradicts the fact that $z^G \cap C_S(Q) = \{z\}$. This proves $W \trianglelefteq S$. Since S centralizes z it follows by order considerations that $|S : C_S(W)| = |S : C_S(z^g)| = 2$, completing the proof. □

The final lemma required to show that $|T| \ngtr 2$ is that $A \trianglelefteq S$. Since $|S : C_S(z^g)| = 2$ we use Sylow's Theorem to arrange $S \cap S^g = C_S(z^g)$ as in Chapter 5.

Lemma 7.14. *For some $u \in N_G(A)$ of 3-power order we have*

$$S \cap S^g = A \rtimes C_{S \cap S^g}(\langle u \rangle), \text{ where } A = [S \cap S^g, u] \text{ and } A \trianglelefteq S.$$

Proof. First we find the required $u \in G$. If we take $s \in Q - Q_0$ and $s_1 \in Q^g - T$ to be involutions then we get $(z^g)^s = zz^g$ and $z^{s_1} = zz^g$, i.e., $u_1 = s_1 s$ normalizes W and induces an automorphism of order 3 on it. Since s and s_1 are both involutions, $\langle s, s_1 \rangle$ is a dihedral group. As an automorphism of W, u_1 has order 3 and as $\langle u_1 \rangle$ is the (unique) maximal cyclic subgroup of $\langle s, s_1 \rangle$, we choose u an element of 3-power order in $\langle u_1 \rangle$ inducing the same automorphism as u_1 on W. Because s and s_1 act on $S \cap S^g$, so too does u. Since this is a $2'$-action, Fitting's Lemma (Lemma 3.11) gives us that

$$S \cap S^g = [S \cap S^g, u]C^* \quad \text{where} \quad C^* = C_{S \cap S^g}(\langle u \rangle) \text{ and } s \text{ inverts } \langle u \rangle.$$

As Q and Q^g are normal in S and S^g respectively, we know that

$$[S \cap S^g, s] \leq Q \cap S^g = Q_0$$

and similarly, $[S \cap S^g, s_1] \leq T$. Now we have that $[S \cap S^g, u] \leq [S \cap S^g, s_1 s] \leq A$. Since $[S \cap S^g, u]$ is abelian, Lemma 3.11 gives us $[S \cap S^g, u] \cap C^* = 1$ which forces $[S \cap S^g, u] = A$. Thus we have shown that $S \cap S^g = A \rtimes C^*$ as desired.

Finally, we observe that we've shown that

$$S = S \cap S^g \langle\, s \,\rangle = (A \rtimes C^*)\langle\, s \,\rangle \text{ with } A = [S \cap S^g, u]$$

and it follows from s inverting u that $A \trianglelefteq S$. □

Up to this point we have focused exclusively on the fusion of elements of Q in S – the component L has not played any role other than to restrict these fusions via the Γ_1-property. But now we see that there is "no room" for L in our structure of S. Let $R = S \cap L \in Syl_2(L)$.

Proof of Theorem 7.1. We have already established this when $|T| = 2$, so we consider when $|T| > 2$. The previous lemma gives us that $A \trianglelefteq S$ and hence $[A, R] \leq A \cap R \leq C_R(Q)$. Since none of the involutions in A centralize Q except z, whereas $z \notin R$ we have

$$[A, R] \leq A \cap R = 1.$$

Now T consists only of outer automorphisms of L as QR contains all the inner automorphisms of L in S. By Lemma 3.17 the involution of field-type is the only outer automorphism of $L_2(q)$ of 2-power order that centralizes R which contradicts the fact that $|T| > 2$. This completes the proof of the theorem. □

Chapter 8

Proof of Theorem 2

In this chapter we consider simple fusion systems with "standard" components of type $L_2(q)$ with centralizers of 2-rank at least 2. Assuming the centralizers of these components also have the Γ_1-embedding property, we characterize such systems.

We recall our hypothesis (see Section 2.5 for a more complete discussion).

Hypothesis 2. Assume $\mathcal{F}$ is a simple saturated fusion system on a 2-group S. Suppose further that $\mathcal{F}$ contains a subgroup Q of rank at least 2 and subsystem $\mathcal{K}$ on R of type $L_2(q)$, such that

(1) $\mathcal{K}$ is a normal component of $C_{\mathcal{F}}(z)$ for every fully $\mathcal{F}$-centralized involution $z \in Z(Q)$,

(2) $C_{\mathcal{C}}(\mathcal{K})$ is a fusion system on Q, where z is any involution as in (1) with $\mathcal{C} = C_{\mathcal{F}}(z)$, and

(3) $C_{\mathcal{C}}(\mathcal{K})$ is Γ_1-embedded for some z.

Our goal in this chapter is to prove the following theorem.

Theorem 8.1. *Under the conditions of Hypothesis 2, $\mathcal{F}$ contains a strongly closed subgroup. In particular, one of the following holds.*

(i) Q is strongly closed, or

(ii) Q is a dihedral subgroup of index 2 contained in a strongly closed dihedral group.

We recall that $\mathcal{F}$ contains a Γ_1-embedded subsystem $\mathcal{Q}$ on a 2-group Q if:

(S1) For any fully $\mathcal{Q}$-centralized involution $w \in Q$, $C_Q(w)$ is strongly closed in $C_{\mathcal{F}}(w)$.

(S2) For any involution $w \in Q$ and $\mathcal{F}$-conjugate $w^\phi \in Q$, there exists a map $\alpha \in \mathrm{Hom}_{\mathcal{Q}}(\langle w \rangle, Q)$ such that $w^\phi = w^\alpha$.

(S3) If $W \leq Q$ is a four-group such that $W^\phi \leq N_S(Q)$ for some $\phi \in \mathrm{Hom}_{\mathcal{F}}(W, N_S(Q))$ then $W^\phi \cap Q \neq 1$.

In proving the theorem we do not explicitly invoke the component subsystem $\mathcal{K}$ or subgroup R until the very last piece of analysis (Theorem 8.18). However, the component is implicit in the Γ_1-property which is essential to the proof throughout.

We will show first that $Q \trianglelefteq S$ and, ultimately, $\mathcal{F}$-conjugates of subgroups of Q will either intersect Q trivially or be contained in Q. Take $t \in S$ such that $\mathcal{Q} = C_C(\mathcal{K})$ is Γ_1-embedded where $C = C_{\mathcal{F}}(t)$.

Lemma 8.2. *$Q \trianglelefteq S$.*

Proof. Suppose Q is not normal in S. Take $s \in N_S(N_S(Q)) - N_S(Q)$ with $s^2 \in N_S(Q)$. Set $T = N_S(Q)\langle s \rangle$ so it follows immediately that $Q \cap Q^s \trianglelefteq T$. By the Γ_1-property $Q \cap Q^s \neq 1$ as $m(Q) \geq 2$. Therefore there exists some involution $z \in Z(T) \cap (Q \cap Q^s)$. Since $z \in Z(Q)$, z is fully $\mathcal{Q}$-centralized, and since $s \in C_S(z)$ we have that $Q^s = Q$ by Condition S1, a contradiction. □

Next we show that conjugates of four-subgroups of Q are actually contained in Q.

Lemma 8.3. *If $W \leq Q$ is a four-group and $W^\phi \leq S$ for some $\phi \in \mathrm{Hom}_{\mathcal{F}}(W, S)$ then $W^\phi \leq Q$.*

Proof. By Alperin's Fusion Theorem there exists a sequence of subgroups of S

$$W = W_0, W_1, ..., W_m = W^\phi \text{ and } V_1, V_2, ..., V_m,$$

with $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(V_i)$, $C_S(V_i) = Z(V_i)$, and

(i) $\phi = \phi_1|_{W_0}\phi_2|_{W_1} \dots \phi_m|_{W_{m-1}}$.

(ii) $W_{i-1}, W_i \leq V_i$, and $W_{i-1}^{\phi_i} = W_i, \quad 1 \leq i \leq m$.

Assume to the contrary that $W^\phi \not\leq Q$. Then there is some smallest $1 \leq i \leq m$ such that $W_i = W_{i-1}^{\phi_i} \not\leq Q$. We may therefore replace W by W_{i-1} and ϕ by ϕ_i. Let $V = V_i$ so that $\phi \in \mathrm{Aut}(V)$.

Since $Q \trianglelefteq S$ we have that V normalizes $Z(Q)$ so that

$$Z(Q) \cap C_S(V) = Z(Q) \cap Z(V) \neq 1.$$

Take an involution $z \in Z(Q) \cap Z(V)$.

By the Γ_1-property, we have that $W^\phi \cap Q \neq 1$ so there is some $w \in W^\sharp$ such that $w^\phi \in Q$. First suppose w^ϕ is fully $\mathcal{Q}$-centralized. We invoke Condition S2 to produce $\alpha \in \mathrm{Hom}_{\mathcal{Q}}(\langle w \rangle, Q)$ such that $w^\phi = w^\alpha$. Since w^ϕ is fully $\mathcal{Q}$-centralized, we can extend α to $C_Q(w) \geq W$. Notice that $\alpha^{-1}\phi$ centralizes w^ϕ so by Condition S1 we get $(W^\alpha)^{\alpha^{-1}\phi} = W^\phi \leq Q$. This contradiction shows w^ϕ is not fully $\mathcal{Q}$-centralized.

There is some $\beta \in \mathrm{Hom}_{\mathcal{Q}}(\langle w \rangle, Q)$ such that w^β is fully $\mathcal{Q}$-centralized. Since w^β is fully $\mathcal{Q}$-centralized we can extend β to $C_Q(w)$ so, in particular, W^β is defined and contained in Q. Replacing ϕ by $\beta^{-1}\phi$ and W by W^β if necessary, we may assume that w is fully $\mathcal{Q}$-centralized and that W^ϕ is not contained in Q. By Condition S2 we can take $\alpha \in \mathrm{Hom}_{\mathcal{Q}}(\langle w^\phi \rangle, Q)$ such that $w^{\phi\alpha} = w$. As before, α extends to $C_Q(w^\phi)$ since w is fully $\mathcal{Q}$-centralized. Also, since $z \in Z(Q)$, z^α is defined (and in Q). Notice that $\alpha^{-1}\phi^{-1}$ centralizes w so by Condition S1 we get

$$(z^{\alpha})^{\alpha^{-1}\phi^{-1}} = z^{\phi^{-1}} \in Q.$$

We reapply Condition S2, this time, taking $\gamma \in \mathrm{Hom}_{\mathcal{Q}}(\langle z^{\phi^{-1}} \rangle, Q)$ such that $z^{\phi^{-1}\gamma} = z$. As z is fully $\mathcal{Q}$-centralized we can extend γ to $C_Q(z^{\phi^{-1}})$. Because ϕ normalizes $Z(V)$, $z^{\phi^{-1}} \in Z(V)$, hence $W \leq C_Q(z^{\phi^{-1}})$, i.e., $W^{\gamma} \leq Q$. Notice that $\gamma^{-1}\phi$ centralizes z so that $(W^{\gamma})^{\gamma^{-1}\phi} = W^{\phi} \leq Q$ by Condition S1. This contradiction completes the proof of the lemma. □

This lemma enables us to show that, as in the group-theoretic case, $\mathcal{F}$-conjugates of subgroups of Q conjugate back into Q or intersect Q trivially.

Lemma 8.4. *If $X \leq Q$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(X, S)$ then either $X^{\phi} \leq Q$ or $X^{\phi} \cap Q = 1$. In the latter case $m(X) \leq 1$.*

Proof. Suppose that $X^{\phi} \cap Q \neq 1$ so, in particular, there is some involution $w \in X$ such that $w^{\phi} \in Q$. Suppose to the contrary that $X^{\phi} \not\leq Q$. We proceed as in Lemma 8.3 and by analogous Alperin Theorem reductions, we may assume $\phi \in \mathrm{Aut}(V)$ where $V \geq X$ has the property that $C_S(V) = Z(V)$. Since $Q \trianglelefteq S$ we have that V normalizes $Z(Q)$ so that $Z(Q) \cap C_S(V) = Z(Q) \cap Z(V) \neq 1$. Take an involution $z \in Z(Q) \cap Z(V)$ and set

$$Y = \langle X, z \rangle \leq Q \cap V.$$

We claim that $z^{\phi} \in Q$. If $z^{\phi} \notin Q$ then $\langle w, z \rangle$ is four-group conjugating out of Q, a contradiction by Lemma 8.3. By Condition S2 we can take $\alpha \in \mathrm{Hom}_{\mathcal{Q}}(\langle z^{\phi} \rangle, S)$ such that $z^{\phi\alpha} = z$. It follows that α extends to $C_Q(z^{\phi})$, since z is fully $\mathcal{Q}$-centralized. Since as before $z^{\phi} \in Z(V)$ it follows that $Y \leq C_Q(z^{\phi})$, so $Y^{\alpha} \leq Q$. Notice that

$\alpha^{-1}\phi^{-1}$ centralizes z, so by Condition S1 we get

$$(Y^{\alpha})^{\alpha^{-1}\phi^{-1}} = Y^{\phi^{-1}} \leq Q.$$

In particular, $z^{\phi^{-1}} \in Q$. We reapply Condition S2, this time taking $\beta \in \mathrm{Hom}_{\mathcal{Q}}(\langle z^{\phi^{-1}} \rangle, Q)$ such that $z^{\phi^{-1}\beta} = z$. As z is fully $\mathcal{Q}$-centralized we can extend β to $C_Q(z^{\phi^{-1}})$. Since $z^{\phi^{-1}} \in Z(V)$ we have that $Y \leq C_Q(z^{\phi^{-1}})$, i.e., $Y^{\beta} \leq Q$. Notice that $\beta^{-1}\phi$ centralizes z so that $(Y^{\beta})^{\beta^{-1}\phi} = Y^{\phi} \leq Q$ by Condition S1. In particular, $X^{\phi} \leq Q$. This contradiction completes the proof. □

Lemma 8.5. *Q is weakly closed in $\mathcal{F}$.*

Proof. This is an immediate consequence of Lemma 8.4. □

For the remainder of this chapter choose $X \leq Q$ to be of maximal order such that there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(X, S)$ with $X^{\phi} \cap Q = 1$. Let $T = X^{\phi}$ and observe that $m(T) = 1$ by Lemma 8.3 and Lemma 8.4. By [Gor80, 5.4.10] this forces T to be cyclic or generalized quaternion.

We first show that if $|T| = 1$ then Q is strongly $\mathcal{F}$-closed. We then prove some basic facts about involutions $u \in Q$ that conjugate out of Q. After this, we complete the proof of Theorem 8.1 by treating the cases where $|T| = 2$ and $|T| > 2$.

Lemma 8.6. *If $T = 1$ then Q is strongly closed in $\mathcal{F}$.*

Proof. This is immediate from Lemma 8.4 and the maximality of T. □

Henceforth we assume that $T \neq 1$ and let $\langle w \rangle = \Omega_1(X)$.

Lemma 8.7. *If $u \in Q$ is an involution such that $u^{\psi} \in S - Q$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle u \rangle, S)$, then u^{ψ} is not fully $\mathcal{F}$-centralized.*

Proof. Suppose to the contrary that u^ψ is fully centralized. Then ψ can be extended to $C_S(u)$ so, in particular, $C_Q(u)^\psi$ is defined. As $u \in C_Q(u)$ and $u^\psi \notin Q$, Lemma 8.4 forces $C_Q(u)^\psi \cap Q = 1$. Because $m(Q) \geq 2$ we claim that $m(C_Q(u)) \geq 2$. If u is not contained in the center of Q then $\langle\, u \,\rangle \times Z(Q) \leq C_Q(u)$ has rank at least 2. If $u \in Z(Q)$ then $C_Q(u) = Q$ which also has rank at least 2. So $C_Q(u)$ contains a four-group, which contradicts our Γ_1-property, thereby completing the proof. □

From this we deduce that any fully centralized conjugate of w lies in Q. For the remainder of the chapter we may therefore assume X is also chosen such that w is fully centralized.

Lemma 8.8. *Let $u \in Q$ be a fully centralized involution such that $u^\psi \in S - Q$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, u \,\rangle, S)$. If $Y \leq C_Q(u^\psi)$ then $m(Y) = 1$.*

Proof. Suppose to the contrary that $m(Y) \geq 2$. Since u is fully centralized we extend ψ^{-1} to $C_S(u^\psi)$ so that $Y^{\psi^{-1}}$ is defined. By Lemma 8.4 and the Γ_1-property we have that $Y^{\psi^{-1}} \leq Q$. But then $\langle\, u, Y^{\psi^{-1}} \,\rangle^\psi \nleq Q$ and $\langle\, u, Y^{\psi^{-1}} \,\rangle^\psi \cap Q \neq 1$ which contradicts Lemma 8.4. Thus we conclude that $m(Y) = 1$. □

8.1 The $|T| = 2$ case

In this section we assume that $|T| = 2$ so $T = \langle\, w^\phi \,\rangle$ for some involution $w \in Q$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, w \,\rangle, S)$. First we show that TQ and Q are both dihedral.

Lemma 8.9. *If $|T| = 2$ then Q is dihedral, TQ is dihedral, and, in particular, $m(Q) = 2$.*

Proof. Suppose that $|C_Q(w^\phi)| > 2$. Since w is fully centralized we can extend ϕ^{-1} to $C_S(w^\phi)$ so, in particular, $Y = C_Q(w^\phi)^{\phi^{-1}}$ is defined. By maximality of T and Lemma 8.4 we have that $Y \leq Q$, so $Z = \langle\, w, Y \,\rangle \leq Q$. But $Z^\phi \nleq Q$ and $Z^\phi \cap Q \neq 1$, a contradiction by Lemma 8.4. We conclude that $|C_Q(w^\phi)| = 2$.

Since $|C_Q(w^\phi)| = 2$, TQ is dihedral or quasidihedral by Suzuki's Lemma. Since Q is a maximal subgroup of $TQ = Q\langle w^\phi \rangle$ and $m(Q) \geq 2$, this forces both Q and TQ to be dihedral, as desired.

□

We now show TQ is strongly closed in $\mathcal{F}$.

Lemma 8.10. *If $|T| = 2$ then $P = TQ$ is a strongly closed dihedral group.*

Proof. Since $P = Q\langle w^\phi \rangle$ is dihedral there is some involution $z \in Q$ such that $Z(P) = \langle z \rangle \leq Z(Q)$. Observe that since Q is normal in S, it follows that $z \in Z(S)$ as well. Since Q is maximal in P we have that all non-central involutions in Q are P-conjugate. Lemma 4.2 gives us that z is not isolated. Hence z conjugates outside Q (in which case we can replace w^ϕ by z^ϕ) or z conjugates into $Q - \langle z \rangle$. In the latter case, we get that z is conjugate to w so we could replace w^ϕ by z^ϕ in this situation as well. So in any case we may assume $P = Q\langle z^\phi \rangle$.

First we show that P is strongly involution closed. Assume not and take $w^\psi \in S - P$ for some fully centralized $w \in Q$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle w \rangle, S)$. If $|C_P(w^\psi)| = 2$ then $P\langle w^\psi \rangle$ must be dihedral by Suzuki's Lemma. However, Q is normal in $P\langle w^\psi \rangle$ and $|P\langle w^\psi \rangle : Q| = 4$ whereas a dihedral group does not have a normal dihedral subgroup of index 4. Thus $|C_P(w^\psi)| > 2$.

If $C_P(w^\psi)$ is cyclic, then there exists an element $x \in C_Q(w^\psi)$ or order 4. Since w is fully centralized, ψ^{-1} extends to $C_S(w^\psi)$ which contains x. By maximality of T it follows that $x^{\psi^{-1}} \in Q$. Let $Y = \langle \langle x^{\psi^{-1}} \rangle, w \rangle \leq Q$ and observe that $x^\psi \in Y^\psi \cap Q \neq 1$ and $w^\psi \notin Q$, a violation of Lemma 8.4. It follows then $C_P(w^\psi)$ contains a four-group which, up to conjugation, contains z^ϕ.

Assume now that $|Q| > 4$. Since w^ψ and z^ϕ both invert the maximal cyclic subgroup of Q it follows that $w^\psi z^\phi$ centralizes that maximal cyclic subgroup of Q.

We now show that $w^{\psi}z^{\phi}$ is conjugate to w. Since $z \in Z(S)$ we can extend ϕ^{-1} to $\langle\, w^{\psi} \,\rangle \leq C_S(z^{\phi})$. We've established that $Q\langle\, w^{\psi\phi^{-1}} \,\rangle$ must be dihedral with center $\langle\, z \,\rangle$, so $w^{\psi\phi^{-1}}$ is conjugate to $w^{\psi\phi^{-1}}z$. It follows at once that w^{ψ} is conjugate to $w^{\psi}z^{\phi}$. Let $Q_0 = C_Q(w^{\psi}z^{\phi})$, the maximal cyclic subgroup of Q, and $w^{\alpha} = w^{\psi}z^{\phi}$ for some $\alpha \in \mathrm{Hom}_{\mathcal{F}}(\langle\, w \,\rangle, S)$. Since w is fully centralized we can extend α^{-1} to $Y = \langle\, w^{\alpha}, Q_0 \,\rangle \leq C_S(w^{\alpha})$. By maximality of T, $Q_0^{\alpha^{-1}} \leq Q$ so that Y is a conjugate of a subgroup of Q which violates Lemma 8.4. Thus P is strongly involution closed when $|Q| > 4$.

Assume, on the other hand, that Q is a four-group. Let $Q = \langle\, u, z \,\rangle$. It follows then that both z^{ϕ} and w^{ψ} conjugate u into uz. Hence $Q = C_Q(w^{\psi}z^{\phi})$. Proceeding as above, we extend ϕ^{-1} to $\langle\, w^{\psi}z^{\phi} \,\rangle$ so that $Q\langle\, w^{\psi\phi^{-1}} \,\rangle$ is dihedral with center $\langle\, z \,\rangle$. As such, we have that $w^{\psi\phi^{-1}}$ is conjugate to $w^{\psi\phi^{-1}}z$. We have found a conjugate of w, say, w^{α} that centralizes Q. Since $C_Q(w^{\alpha}) = Q$ has rank 2, we have a contradiction by Lemma 8.8. Thus P is strongly involution closed in this case as well.

Now we show that P is strongly closed (no matter the order of Q). Take $y \in P$ of order 4 or more such that $y^{\psi} \in S - P$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle\, y \,\rangle, S)$. By maximality of T it follows that $y \notin Q$ which forces $\langle\, y \,\rangle$ to be the maximal cyclic subgroup of P. Thus $y^2 \in Q$, $\langle\, y^2 \,\rangle = \Phi(P)$, and

$$\langle\, (y^2)^{\psi} \,\rangle = \langle\, y^2 \,\rangle = \Phi(P), \text{ that is, } \psi \in N_{\mathcal{F}}(\Phi(P)) \leq C_{\mathcal{F}}(z).$$

Now let .

$$N_{\mathcal{F}}(\Phi(P))/\Phi(P) = \overline{N_{\mathcal{F}}(\Phi(P))}.$$

Note that $\overline{P} = \langle\, \overline{Q} \,\rangle \times \langle\, \overline{z^{\phi}} \,\rangle$. We will show that $\overline{P}$ is strongly closed in $\overline{S}$ with respect to $\overline{N_{\mathcal{F}}(\Phi(P))}$, which is sufficient to complete the proof as $\psi \in N_{\mathcal{F}}(\Phi(P))$.

Notice that the coset $\overline{Qz^{\phi}} = \overline{y}$ does not contain an involution. This fact, that

Q is strongly closed in $C_{\mathcal{F}}(z) \geq N_{\mathcal{F}}(\Phi(P))$, and that P is strongly involution closed together forces $\langle \overline{z^\phi} \rangle$ to be strongly closed as well. By Lemma 4.1 we conclude that $\overline{P}$ is strongly closed in $\overline{N_{\mathcal{F}}(\Phi(P))}$ and hence P is strongly $\mathcal{F}$-closed as well. $\square$

We've shown Theorem 8.1 to be true when $|T| = 2$ so we henceforth consider when $|T| > 2$. As in the previous chapter, this forces $|Q| \geq 8$.

8.2 The $|T| > 2$ case

Lemma 8.11. *Let $x^\phi \in T$ be an element of order 4 (so $(x^\phi)^2 = w^\phi$). Set $P = \langle x^\phi \rangle Q$. Then*

(i) Q does not contain a four-group normalized by P,

(ii) $|Q : C_Q(w^\phi)| = 2$, and

(iii) $m(Q) = 2$.

Proof. We first show that P contains a normal four-group. Suppose the contrary so that P is a dihedral or quasidihedral group by [Gor80, 5.4.10]. In either case P contains the normal subgroup Q of index 4. This is contrary to the fact that the cyclic group $\Phi(P)$ is the unique normal subgroup of index 4 in any dihedral or quasidihedral group of order at least 2^4. Thus there exists a normal four-group U in P.

Note that $w^\phi = (x^\phi)^2$ centralizes U. If $U \leq Q$ then since w is fully centralized we can extend ϕ^{-1} to $C_Q(w^\phi) \geq U$. This contradicts Lemma 8.8 so we conclude that $U \not\leq Q$. Since U cannot be chosen to lie in Q we have that $Z(Q)$ must be cyclic. Let $\Omega_1(Z(Q)) = \langle z \rangle$ so that $z \in U$. It follows at once that $U = \langle w^\phi q, z \rangle$ for some $q \in Q$. Since w^ϕ centralizes U, $[w^\phi, q] = 1$ and $q^2 = 1$. Since w^ϕ centralizes $\langle q, z \rangle$ but does not centralize any four-group in Q, $q = 1$ or $q = z$; in either case,

$$U = \langle w^\phi, z \rangle.$$

It follows that $|P : C_P(w^\phi)| = 2$, because if this were index 1, we would have $[Q, w^\phi] = 1$. This fact would, after extending ϕ^{-1}, force $(\langle w^\phi \rangle Q)^{\phi^{-1}} = Q$, a contradiction. In particular, we have that $|Q : C_Q(w^\phi)| = 2$.

If $m(Q) \geq 3$ then Q has a subgroup $E \cong E_8$ and so $E \cap C_P(w^\phi)$ contains a four-group. By the same argument as in the first paragraph, we obtain a four-subgroup of Q which conjugates out of Q, a contradiction. Thus $m(Q) = 2$.

□

For the remainder of this section preserve the notation from the previous lemma. We now show that Q is dihedral.

Lemma 8.12. *Q is a dihedral group of order 2^n with $n \geq 3$, T is a cyclic group of order 2^{n-1}, and TQ is a wreathed product.*

Proof. We first show that T is abelian and hence cyclic since $m(T) = 1$. If Q normalizes T then $[Q, T] \leq Q \cap T = 1$ so $Q \leq C_S(w^\phi)$, a contradiction by Lemma 8.8. Thus there exists some $s \in Q$ such that

$$s \in N_{TQ}(N_{TQ}(T)) - N_{TQ}(T), \text{ so } T^s \neq T.$$

Since T and T^s normalize each other we have that $[T, T^s] \leq T \cap T^s$. If $w^\phi \in T \cap T^s$ then $TT^s \leq C_S(w^\phi)$. Thus we can extend ϕ^{-1} to TT^s. Since $T^{\phi^{-1}} = X \leq Q$, Lemma 8.4 implies that $(TT^s)^{\phi^{-1}} \leq Q$. By maximality of T, this forces $T = T^s$, a contradiction. We conclude that $T \cap T^s = 1$ and so $[T, T^s] = 1$. By the previous lemma, $|Q : C_Q(w^\phi)| = 2$ which by maximality forces X to be of index 2 in Q. It follows that

$$|TQ : T \times T^s| = 2.$$

Let $Q_0 = Q \cap (T \times T^s)$ so that because $[T, s] \leq Q_0$ we have

$$T \times T^s = T \times Q_0 = T^s \times Q_0.$$

Since $[Q_0, T] = 1$ and $[T^s, T] = 1$ it follows that $T \leq Z(T \times T^s)$. Therefore T is abelian and hence cyclic.

By [Gor80, 5.4.4] Q must be a modular, dihedral, or quasidihedral group. If Q is modular, then $\Omega_1(Q)$ is a characteristic four-group; but we established in the previous lemma Q may not posses such a group. Thus we conclude that Q is dihedral or quasidihedral.

Finally, that proof that $TQ \cong Z_{2^{n-1}} \wr Z_2$ is identical to Lemma 7.9, and Q is then dihedral by Lemma 3.8. □

We've established that $w^\phi = z^\phi$ where $Z(Q) = \langle z \rangle$. Preserving this notation, we prove several useful properties of conjugates in $z^{\mathcal{F}}$.

Lemma 8.13. $z^{\mathcal{F}} \cap C_S(Q) = \langle z \rangle$.

Proof. This is immediate by Lemma 8.8. □

Lemma 8.14. *For any two distinct commuting $\mathcal{F}$-conjugates z_1,z_2 of z, $z_1 z_2$ is also $\mathcal{F}$-conjugate to z. Furthermore, if $z = z_1$ then z_2 may be conjugated into zz_2 by some element $q \in Q$.*

Proof. We may assume $z_1 = z$ and $z_2 = z^\psi$ for some $\psi \in \mathrm{Hom}_{\mathcal{F}}(\langle z \rangle, S)$. Let $Y = \langle z^\psi \rangle Q$, $U = \langle z, z^\psi \rangle$, and $C^* = C_Y(z^\psi) = C_Y(U)$. First, we observe that $Q \not\leq C^*$ by Lemma 8.13 and hence $C^* < C_Y(z^\psi)$. Now we claim that U is the four-subgroup of C^*. If there were another four-group $U_1 \neq U$ then $\langle U, U_1 \rangle$ would contain a subgroup $E \cong E_8$. Then $E \cap Q$ has rank at least two and centralizes z^ψ which contradicts Lemma 8.8. We have that $U \operatorname{char} C^*$ and therefore $N_Y(C^*)$ acts on U. It follows at once that for any $y \in N_Y(C^*) - C^*$ we get $(z^\psi)^y = zz^\psi$. If $y = z^\psi q$ for some $q \in Q$ then $(z^\psi)^q = zz^\psi$ so the desired conjugation may be effected by an element of Q, completing the proof. $\square$

Lemma 8.15. *$W = \langle\, z, z^\phi \,\rangle$ is normal in S, and so $|S : C_S(z^\phi)| = 2$.*

Proof. Let $A = TQ_0$ where Q_0 is the maximal cyclic subgroup of Q. If W is not normal in S then neither is A as $W = \Omega_1(A)$ is characteristic in A. Suppose this is the case and take $u \in N_S(N_S(A)) - N_S(A)$ so

$$A^u \leq N_S(A) = N_S(A^u) = N_S(A)^u.$$

We first suppose that $W^u = W$. This forces $z^\phi \in W^u$ so that $(z^\phi)^u = z^\phi$ or $(z^\phi)^u = zz^\phi$. Since $TQ = A\langle\, s \,\rangle \cong Z_{2^{n-1}} \wr Z_2$ for an involution $s \in Q$, we can replace u by us if necessary to get $(z^\phi)^u = z^\phi$ so $u \in C_S(z^\phi)$. We now extend ϕ^{-1} to a map $\phi_1^{-1} \in \mathrm{Hom}_{\mathcal{F}}(C_S(z^\phi), S)$ such that $z^{\phi\phi_1^{-1}} = z$ (we write ϕ_1^{-1} for the extension of ϕ^{-1} to $C_S(z^\phi)$ to differentiate from the inverse of the extension of ϕ to $C_S(z^{\phi^{-1}})$). Since $\phi\phi_1^{-1} \in C_{\mathcal{F}}(\langle\, z \,\rangle)$ and Q is strongly closed in $C_{\mathcal{F}}(\langle\, z \,\rangle)$ (by Condition S1) we get that

$$T^{\phi_1^{-1}} = Q_0^{\phi\phi_1^{-1}} \leq Q. \tag{8.1}$$

The maximality of T forces $T^{\phi_1^{-1}} = Q_0$. Since $u^{\phi_1^{-1}}$ normalizes Q_0, it follows that u normalizes $Q_0^{\phi_1} = T$. As u normalizes Q_0 as well, it follows that $u \in N_S(A)$, a contradiction. This shows that $W^u \neq W$.

Next, notice that $[A, A^u] \leq A \cap A^u = Q_0$ and also

$$T \leq N_S(A) = N_S(A^u) \leq N_S(W^u).$$

This forces $[(z^{\phi})^u, T] \leq W^u$ and hence

$$[(z^{\phi})^u, T] \leq Q_0 \cap W^u = \langle z \rangle.$$

Now let $Y = \langle (z^{\phi})^u, z, T \rangle$ so that

$$\overline{Y} = Y/\langle z \rangle \cong Z_2 \times Z_{2^{n-1}}.$$

By Lemma 3.6 we have that $\overline{Y}$ cannot act faithfully on Q. Thus at least one of $\overline{z^{\phi}}$, $\overline{(z^{\phi})^u}$, or $\overline{z^{\phi}(z^{\phi})^u}$ centralizes Q. Since z^{ϕ} is a square in T we have that z^{ϕ} centralizes W^u and, in particular, $[z^{\phi}, (z^{\phi})^u] = 1$. By Lemma 8.14, z is $\mathcal{F}$-conjugate to $z^{\phi}(z^{\phi})^u$ so no matter which involution in $\overline{X}$ centralizes Q we have a contradiction as $z^{\mathcal{F}} \cap C_S(Q) = \{z\}$ by Lemma 8.13. Since S commutes with z it follows by order considerations that $|S : C_S(W)| = |S : C_S(z^{\phi})| = 2$. □

We remark that as $|S : C_S(z^{\phi})| = 2$ we can view ϕ as an isomorphism from $C_S(z^{\phi^{-1}})$ into $C_S(z^{\phi})$. Now we determine the structure of $C_S(z^{\phi})$.

Lemma 8.16. *Let $A = TQ_0 \cong Z_{2^{n-1}} \times Z_{2^{n-1}}$. Then there is some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(z^{\phi}))$ of 3-power order such that*

$$C_S(z^{\phi}) = A \rtimes C_{C_S(z^{\phi})}(\langle \alpha \rangle) \text{ where } A = [C_S(z^{\phi}), \alpha] \text{ and } A \trianglelefteq S.$$

Proof. First we find the required $\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(z^{\phi}))$. For $s \in Q - Q_0$ as above we have that $z^{\phi c_s} = zz^{\phi}$ and $z^{c_s} = z$ so c_s is an involution in $\mathrm{Aut}_S(W)$. By Lemma 8.14 there is some $s' \in Q$ such that $z^{\phi^{-1} c_{s'}} = zz^{\phi^{-1}}$. So $c^*_{s'} = \phi^{-1} c_{s'} \phi \in \mathrm{Aut}_{\mathcal{F}}(C_S(z^{\phi}))$ is another involution in $\mathrm{Aut}_{\mathcal{F}}(W)$. Thus $\alpha = c_s c^*_{s'}$ has order 3 as an automorphism of W, and order divisible by 3 as an automorphism of $C_S(z^{\phi})$. We may replace α by an appropriate power so that α is of 3-power order (as an automorphism of $C_S(z^\phi)$). Since α has a $2'$ action on $C_S(z^\phi)$, Fitting's Lemma (Lemma 3.11) gives us that

$$C_S(z^\phi) = [C_S(z^\phi), \alpha]C, \text{ where } C = C_{C_S(z^\phi)}(\langle\, \alpha \,\rangle).$$

We first show that $[C_S(z^\phi), \alpha] \leq A$. Since $s \in Q$ we know

$$[C_S(z^\phi), c_s] = [C_S(z^\phi), s] \leq C_S(z^\phi) \cap Q = Q_0.$$

Now, take $g \in C_S(z^\phi)$ so then

$$\begin{aligned} [g, c^*_{s'}] = g^{-1}g^{c^*_{s'}} &= g^{-1}g^{\phi^{-1}c_{s'}\phi} \\ &= ((g^{-1})^{\phi^{-1}} g^{\phi^{-1}c_{s'}})^\phi \\ &= [g^{\phi^{-1}}, c_{s'}]^\phi \in (C_S(z^{\phi^{-1}}) \cap Q)^\phi = T. \end{aligned}$$

It follows then from the above that $[C_S(z^\phi), c^*_{s'}] \leq T$. Observe $c^*_{s'} \in C_{\mathcal{F}}(z)$ so we get $Q_0^{c^*_{s'}} = Q_0$ by Condition S1. Since

$$[g, \alpha] = [g, c_s c^*_{s'}] = [g, c^*_{s'}][g, c_s]^{c^*_{s'}} \in TQ_0^{c^*_{s'}} = A \qquad \text{for all } g \in C_S(z^\phi)$$

we conclude that $[C_S(z^\phi), \alpha] \leq TQ_0 = A$. Since α acts nontrivially on the four-group $W = \Omega_1(A)$, we have $C \cap A = 1$, and $[C_S(z^\phi), \alpha] = A$. Thus $C_S(z^\phi) = A \rtimes C$ as desired.

Finally, we observe that we have shown that

$$S = C_S(z^\phi)\langle\, s \,\rangle = (A \rtimes C_{C_S(z^\phi)}(\langle\, \alpha \,\rangle))\langle\, s \,\rangle \text{ with } A = [C_S(z^\phi), \alpha].$$

From this it follows that $A \trianglelefteq S$ since c_s inverts α. This completes the proof. $\square$

Continuing with the notation of $C = C_{C_S(z^\phi)}(\langle \alpha \rangle)$ we remark that C acts as automorphisms of the maximal cyclic subgroups of A by Lemma 3.15 and so in particular for $c \in C$ we have that $c^{2^{n-2}}$ centralizes A.

Lemma 8.17. *If $Y \in Q_0^{\mathcal{F}}$ then $\Omega_1(Y) \leq W$.*

Proof. Take $\beta \in \mathrm{Hom}_{\mathcal{F}}(Q_0, S)$ and let $\langle x \rangle = Q_0$. We note that since $Q \trianglelefteq S$ and s normalizes C

$$[C, s] \leq C \cap Q = 1.$$

Write $x^\beta = gcs^\epsilon$ where $g \in A$, $c \in C$, and $\epsilon = 0$ or 1. It follows then from Lemma 3.15 that

$$(x^\beta)^{2^{n-2}} = z^\beta = g_0 c^{2^{n-2}} = g_0 c_0 \text{ for some } g_0 \in A \text{ and } c_0 \in C_C(A).$$

Since c_0 commutes with g_0 and g_0c_0 has order 2, $g_0 \in W$. Suppose $c_0 \neq 1$ and choose i so that $g_0^{\alpha^i} = 1$ or z. Since $c_0 \in C$, α^i fixes c_0 and it follows that $(g_0c_0)^{\alpha^i} = c_0$ or zc_0. In either case we have

$$\langle z, s \rangle \leq C_Q((g_0c_0)^{\alpha^i}).$$

But $(g_0c_0)^{\alpha^i} = z^{\beta\alpha^i}$ which contradicts Lemma 8.8. Thus $c_0 = 1$ and we conclude that $z^\beta \in W$, as needed to prove the lemma. $\square$

We now complete the proof of Theorem 8.1. We also reiterate that the component fusion system $\mathcal{K}$ of type $L_2(q)$ on its dihedral 2-group $R \leq S$ has not been explicitly needed in the preceding arguments (it is implicit in the Γ_1-hypothesis). It is for this final stage of the proof only that we require properties of $\mathcal{K}$ beyond the Γ_1-property.

Theorem 8.18. *Theorem 2 holds, that is, there is no fusion system $\mathcal{F}$ satisfying our hypothesis with $|T| > 2$.*

Proof. Lemma 8.16 gives us that $A \trianglelefteq S$ and hence $[A, R] \leq A \cap R \leq C_R(Q)$. Since none of the involutions in A centralize Q except z, whereas $z \notin R$ we have

$$[A, R] \leq A \cap R = 1.$$

In particular, we have that $[T, R] = 1$. If we can mimic the result of Lemma 3.17 (5) and show that $z^\phi \in C_S(\mathcal{K}) = Q$ we arrive at a contradiction, i.e. $\mathcal{F}$ does not exist.

Since we have just shown that $z^\phi \in C_S(R)$, to show that $z^\phi \in Q$ it remains to prove that $\mathcal{K} \subseteq C_\mathcal{F}(z^\phi)$. By Alperin's Fusion Theorem it suffices to show the following: For each $V \in \mathcal{K}^{fcr}$ and all $\psi \in \mathrm{Aut}_\mathcal{K}(V)$, that ψ extends to $V\langle z^\phi \rangle$ and satisfies $(z^\phi)^\psi = z^\phi$.

All subgroups of R are cyclic or dihedral and hence all but the four-groups among them have 2-groups for automorphism groups. Suppose $V \in \mathcal{K}^{fcr}$ is not a four-group, so $\mathrm{Aut}_\mathcal{K}(V) = \mathrm{Aut}_R(V)$ by the Sylow Axiom for saturated fusion systems. It follows that for each $\psi \in \mathrm{Aut}_\mathcal{K}(V)$ we have $\psi = c_g$ for some $g \in R$. As R centralizes z^ϕ, c_g extends to $V\langle z^\phi \rangle$ with $(z^\phi)^{c_g} = (z^\phi)^g = 1$. Hence to show $\mathcal{K} \subseteq C_\mathcal{F}(z^\phi)$ we need only consider the four-groups of $\mathcal{K}$, all of which are in $\mathcal{K}^{fcr}$. Since $\mathcal{K}$ is the fusion system of $L_2(q)$, by the Example in Section 2.2, $\mathcal{K}$ has two $\mathcal{K}$-classes of four-groups, and all four-groups have automorphism groups isomorphic to S_3. The arguments that follow are therefore independent of the class from which we choose a four-group.

Take a four-group $V \leq R$ so that $\mathrm{Aut}_\mathcal{K}(V) \cong S_3$. By Lemma 8.5, Q is weakly closed, which forces $(VQ)^\beta = V^\beta Q$ for any $\beta \in \mathrm{Hom}_\mathcal{F}(VQ, S)$. Since by hypothesis R is strongly closed in $C_\mathcal{F}(z)$ and β maps $Z(Q) = \langle z \rangle$ to itself, this forces $V^\beta \in \mathcal{K}$. Since there are two $\mathcal{K}$-classes of four-groups we may suppose without loss of generality that we have chosen V such that VQ is fully $\mathcal{F}$-centralized.

We next show that each map in $\mathrm{Aut}_{\mathcal{K}}(V)$ extends to a map on $VQ\langle z^{\phi}\rangle$ that centralizes $Q\langle z^{\phi}\rangle$. Suppose $\psi \in \mathrm{Aut}_{\mathcal{K}}(V)$ is an element of order 3. Since $\mathcal{K} \subseteq C_{\mathcal{F}}(Q)$ we have that ψ extends to VQ such that ψ is the identity on Q, so $\psi \in \mathrm{Aut}_{\mathcal{F}}(VQ)$. As VQ is fully $\mathcal{F}$-centralized we can extend ψ further yet to N_{ψ}, and we claim $T \leq N_{\psi}$ where we recall that

$$N_{\psi} = \{g \in N_S(VQ) \mid \psi^{-1}c_g\psi \in \mathrm{Aut}_S(VQ)\}.$$

Let $\langle y\rangle = T$. We know that y normalizes Q, and since ψ is the identity on Q we have that $\psi^{-1}c_y\psi = c_y$ on Q. But y centralizes $V \leq R$ so that $\psi^{-1}c_y\psi$ is the identity on V. Therefore we have that $\psi^{-1}c_y\psi = c_y$ and so $y \in N_{\psi}$, thereby proving our claim about T. By Lemma 8.17, $(z^{\phi})^{\psi} \in \Omega_1(A)$ so that ψ is an automorphism of $VQ\langle z^{\phi}\rangle$; and we may replace this extended ψ by an appropriate power so that ψ has 3-power order. We know $z^{\psi} = z$, so if $(z^{\phi})^{\psi} \neq z^{\phi}$ then ψ induces an automorphism of $\Omega_1(A)$ of order 2; this contradicts the order of ψ as an automorphism of $VQ\langle z^{\phi}\rangle$. Therefore ψ centralizes z^{ϕ}.

Finally take $g \in R$ such that c_g is an automorphism of V of order 2. The element g commutes both with Q and T so we have at once that c_g extends to a map centralizing z^{ϕ}. As $\mathrm{Aut}_{\mathcal{K}}(V) = \langle \psi, c_g\rangle$ we have shown that each map in $\mathrm{Aut}_{\mathcal{K}}(V)$ extends to a map centralizing z^{ϕ}.

For any $\psi \in \mathrm{Aut}_{\mathcal{K}}(V)$ we have shown that ψ extends to a map that centralizes $\Omega_1(A)$. So consider an arbitrary conjugate of V in $\mathcal{K}$, i.e. take $V^{\beta} \in V^{\mathcal{F}}$ for some $\beta \in \mathrm{Hom}_{\mathcal{K}}(V, V^{\beta})$. As $\mathcal{K} \subseteq C_{\mathcal{F}}(Q)$ we can extend β to VQ such that β is the identity on Q. Since VQ is fully $\mathcal{F}$-centralized, we can extend β^{-1} to $N_{\beta^{-1}}$. Arguing as above, we have that $T \leq N_{\beta^{-1}}$ so β^{-1} extends to a map on T.

Since $\mathrm{Aut}_{\mathcal{K}}(V^\beta) = \mathrm{Aut}_{\mathcal{K}}(V)^\beta$ we have that each map in $\mathrm{Aut}_{\mathcal{K}}(V^\beta)$ is of the form $\beta^{-1}\psi\beta$ for $\psi \in \mathrm{Aut}_{\mathcal{K}}(V)$. We know that β^{-1} extends to $V^\beta Q\langle\, z^\phi \,\rangle$ and we have already shown ψ extends to $VQ\langle\, z^\phi \,\rangle$ such that $(z^\phi)^\psi = z^\phi$. By Lemma 8.17 (which applies to T by the previous paragraph) we have that $(z^\phi)^{\beta^{-1}} \in \Omega_1(A)$ so that ψ centralizes $(z^\phi)^{\beta^{-1}}$. Thus $\beta^{-1}\psi\beta$ centralizes z^ϕ yielding the desired extension of an arbitrary map in $\mathrm{Aut}_{\mathcal{K}}(V^\beta)$. This shows z^ϕ centralizes $\mathcal{K}$ and so $z^\phi \in C_S(\mathcal{K}) = Q$, a contradiction. This completes the proof of the theorem. □

Chapter 9

Future Work

In this thesis we have characterized, to a large extent, simple fusion systems with standard components of type $SL_2(q)$ and $L_2(q)$. However, there are still issues we would like to resolve and results that we would like to sharpen. To that end, we briefly discuss these questions and some potential paths forward in this chapter.

9.1 Further Research I

In Chapter 6 we determined that if $\mathcal{F}$ is a simple saturated fusion system on a 2-group S with standard component of type $SL_2(q)$, $q > 9$, odd then either $\mathcal{F}$ is the system of $G_2(q_1)$ or $L_3(q_1)$ for suitable odd q_1 *or* $\mathcal{F}$ contains a strongly closed quasidihedral subgroup P. We would like to sharpen this last conclusion by actually attaching a normal *subsystem* $\mathcal{E}$ to this strongly closed subgroup (thereby contradicting the simplicity of $\mathcal{F}$). In particular, as all involutions in P are conjugate, we would like to show that $\mathcal{E}$ is the fusion system of $L_3(q_1)$ for suitable odd q_1. On the other hand, if we simply focus on seeking out normal subsystems there are tools at our disposal. For example, we point out that it is necessary only to produce a *weakly* normal subsystem $\mathcal{E}$ of $\mathcal{F}$ thanks to a result due to Craven ([Cra11a]):

Theorem 9.1. *Let $\mathcal{F}$ be a saturated fusion system and $\mathcal{E}$ a weakly normal subsystem of $\mathcal{F}$. Then $O^{p'}(\mathcal{E})$ is a normal subsystem of $\mathcal{F}$.*

In the above theorem $O^{p'}(\mathcal{E})$ is the minimal normal fusion subsystem of "index prime to p". This has the immediate corollary that a saturated fusion system contains a normal subsystem if and only if it contains a weakly normal subsystem.

In Section 7 of [Asc08] Aschbacher develops a process by which one begins with a strongly closed subgroup and constructs a potentially normal subsystem via what he calls *normal maps*. In particular, in [Asc08, Theorem 3] he shows that a subsystem $\mathcal{E}$ on T is normal if and only if there is a normal map on T (with an additional property). In light of Theorem 9.1, one could even restrict to consideration of *weakly normal* maps.

Another approach to treating our strongly closed quasidihedral subgroup in a saturated fusion system is to prove a theorem analogous to that of Hall ([Hal76]). In this paper he classified all groups G such that $O^{2'}(G) = G$ and G contains a strongly closed dihedral 2-subgroup. We also point out that one of the conclusions of our $L_2(q)$ problem involves a strongly closed dihedral group. So if we could prove a fusion systems version of the original theorem then that would treat one of the remaining cases in the $L_2(q)$ problem. Ostensibly, similar methods could be employed in proving these theorems.

9.2 Further Research II

In Theorem 1 our second hypothesis is that the generalized quaternion group R is strongly closed in $C_{\mathcal{F}}(y)$ whenever y is an involution in $C_S(R)$. However, with the exception of Lemma 6.10, we only used that R was strongly closed in $C_{\mathcal{F}}(z)$ where $\langle z \rangle = Z(R)$. So one improvement we might make to Theorem 1 is to reduce the second hypothesis to that R is strongly closed in $C_{\mathcal{F}}(z)$. We have not pursued this yet because our ultimate aim is to remove the second hypothesis altogether, i.e., prove the following conjecture.

Conjecture. Theorem 1 remains true if we remove the second hypothesis altogether.

One approach at this point is to prove this inductively: assuming only that R is weakly $\mathcal{F}$-closed in S, consider any involution $y \in C_S(R)$. Then either $y \in Z(\mathcal{F})$ – which would not happen if we imposed $O_2(\mathcal{F}) = 1$ – or $C_{\mathcal{F}}(y)$ is a "smaller" fusion system to which we might apply induction. In particular, our Theorem 1 would (inductively) tell us about the "normal closure" of R in $C_{\mathcal{F}}(y)$ – it should be a component of type $L_3(q)$ or $G_2(q)$.

9.3 Further Research III

As alluded to in Section 9.1, we still have one remaining unclassified case from our $L_2(q)$ work: when Q, the Sylow 2-subgroup of the centralizer of the component, is strongly closed. Here we see two possible paths. The first is to assume the whole of Aschbacher's definition of tightly embedded on Q and quote a (unpublished draft) result of his in [Asc11b]. His result gives that Q is elementary abelian and hence $\mathcal{F}_Q(Q)$ would be a normal subsystem as Q is strongly closed *or* the fusion system on Q is subnormal (so $\mathcal{F}$ is not simple). The clear downside to this approach is that we would have to take on additional hypotheses.

Alternatively, we strive to prove a fusion system theorem analogous to the following group-theoretic result due to Goldschmidt.

Theorem 9.2 ([Gol75], Corollary B1)**.** *Suppose S is a strongly closed 2-subgroup of any finite group G. Then $C_G(S)^{(\infty)}O_{2'}(G) \trianglelefteq G$.*

In a fusion systems version of this theorem $O_{2'}(G)$ would fall away as it contributes nothing to the fusion on a Sylow 2-subgroup of G. In this context our standard component $\mathcal{K}$ centralizes the strongly closed 2-subgroup Q, which would force $\mathcal{K}$ (or a subsystem containing it) to be normal in $\mathcal{F}$. Generalizations of this result or Hall's result would certainly be contributions to study of fusion systems in general, not just to our standard form problems. Moreover, this result would contribute to our proposed research in Section 9.1.

9.4 Further Research IV

Finally, and most importantly, in our $L_2(q)$ problem we would like show that our standard form hypothesis implies the Γ_1-property, and so this "axiom" for our setup would become extraneous. The reason for such a pursuit is a lacuna in our work when $q \leq 9$. In fact only $q = 9$ and 7 occur (because when $q = 5$ fusion systems components are not quasisimple, so do not arise). In both these cases $L_2(q)$ has D_8 Sylow 2-subgroups, so our Theorem 2 ostensibly covers all $L_2(q)$ fusion systems standard form problems when a Sylow 2-subgroup of $L_2(q)$ has order at least 16. We note in Section 2.5, however, that in the presence of the Γ_1-property we can actually *weaken* the condition of fusion system tight embedding of the centralizer of our component, so there is some positive trade-off in our $L_2(q)$ fusion systems hypotheses.

One of the main obstructions is that it is not clear what the definition of $\Gamma_{U,1}(\mathcal{K})$ should be for a four-group U and component $\mathcal{K}$. One approach we might follow is established by Lynd in his thesis [Lyn12]. Here he appeals to the theory of linking systems to embed the Sylow 2-group of an extension of $\mathcal{K}$ into $\mathrm{Aut}(L_2(q))$. At this point, we have a handle on the structure of the involution centralizer (of which $\mathcal{K}$ is a component) so that we might then invoke the *group-theoretic* Γ_1-property and possibly mimic our arguments from Lemma 7.3.

Bibliography

[AKO11] M. Aschbacher, R. Kessar, and B. Oliver, *Fusion Systems in algebra and toplogy*, London Mathematical Society Lecture Notes Series, vol. 391, Cambridge University Press, 2011.

[Asc75] M. Aschbacher, *On finite groups of component type*, Illiois J. Math. **19** (1975), 78–115.

[Asc77a] ———, *A characterization of Chevalley groups over fields of odd order, I and II*, Ann. of Math. **106** (1977), 353–398.

[Asc77b] ———, *A characterization of Chevalley groups over fields of odd order, II*, Ann. of Math. **106** (1977), 399–468.

[Asc00] ———, *Finite Group Theory*, second ed., Cambridge studies in advanced mathematics, vol. 10, Cambridge University Press, 2000.

[Asc08] ———, *Normal subsystems of fusion systems*, Proc. of Lond. Math. Soc. **97** (2008), no. 1, 239–271.

[Asc11a] ———, *The generalized Fitting subsystem of a fusion system*, Mem. Amer. Math. Soc. **209** (2011), no. 986, v+110.

[Asc11b] ———, *Tightly embedded subsystems of fusion systems*, Preprint (2011).

[Ben94] D. Benson, *Conway's group Co_3 and the Dickson invariants*, Manuscripts Math. **85** (1994), 177–193.

[BLO03] C. Broto, R. Levi, and B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856.

[Cle06] M. Clelland, *Saturated fusion systems and finite groups*, Ph.D. thesis, The University of Birmingham, 2006.

[CP10] M. Clelland and C. Parker, *Two families of exotic fusion systems*, J. Algebra **323** (2010), 287–304.

[Cra11a] D. Craven, *Normal subsystems of fusion systems*, J. Lond. Math. Soc. **84** (2011), 137–158.

[Cra11b] ______, *The Theory of Fusion Systems*, Cambridge studies in advanced mathematics, vol. 131, Cambridge University Press, 2011.

[DF04] D. Dummit and R. Foote, *Abstract Algebra*, third ed., John Wiley and Sons, 2004.

[Foo76a] R. Foote, *Finite groups with components of* 2*-rank 1, I*, J. Algebra **41** (1976), no. 1, 16–46.

[Foo76b] ______, *Finite groups with components of* 2*-rank 1,II*, J. Algebra **41** (1976), no. 1, 47–57.

[Foo78] ______, *Finite groups with maximal* 2*-components of type* $L_2(q)$*, q odd*, Proc. London. Math. Soc. **37** (1978), no. 3, 422–458.

[GH71] D. Gorenstein and K. Harada, *Finite simple groups of low 2-rank and the families* $G_2(q), D_4^2(q)$*, q odd*, Bull. Amer. Math. Soc. **77** (1971), 829–862.

[Gla66] G. Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420.

[GLS96] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of finite simple groups, number 2*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, 1996.

[GLS05] ______, *The classification of finite simple groups, number 6*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, 2005.

[Gol75] D. Goldschmidt, *Strongly closed* 2*-subgroups of finite groups*, Ann. Math. **102** (1975), 475–489.

[Gor80] D. Gorenstein, *Finite Groups*, second ed., AMS Chelsea, 1980.

[Hal76] J.I. Hall, *Fusion and dihedral* 2*-subgroups*, J. Algebra **40** (1976), 203–228.

[Lin06] M. Linckelmann, *Simple fusion systems and the Solomon* 2*-local groups*, J. Algebra **296** (2006), 385–401.

[LO02] R. Levi and B. Oliver, *Construction of* 2*-local finite groups of a type studied by Solomon and Benson*, Geom. Topol. **6** (2002), 917–990.

[Lyn12] J. Lynd, *A characterization of the* 2*-fusion system of* $L_4(q)$, Ph.D. thesis, The Ohio State University, 2012.

[Oli10] B. Oliver, *Extensions of linking systems and fusion systems*, Trans. Amer. Math. Soc. **362** (2010), 5483–5500.

[Oli11] ______, *Reduced fusion systems over* 2*-groups of sectional rank at most four*, Preprint (2011).

[Pui06] L. Puig, *Frobenius categories*, J. Algebra **303** (2006), no. 1, 309–357.

[RS09] K. Roberts and S. Shpectorov, *On the definition of saturated fusion systems*, J. Group Theory **12** (2009), 679–687.

[RV04] A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. **248** (2004), 45–65.

[Sol74] R. Solomon, *Finite groups with Sylow 2-subgroups of type .3*, J. Algebra **28** (1974), 182–198.

[Sta06] R. Stancu, *Control of fusion in fusion systems*, J. Algebra Appl. **5** (2006), 817–837.

[Suz51] M. Suzuki, *A characterization of simple groups $LF(2, p)$*, J. Fac. Sci. Univ. Tokyo. Sect. I. **6** (1951), 259–293.