Passing at the PhD level is accomplished by solving at least two problems in each section while either solving a total of 6 problems or solving 5 problems and making significant progress on two others.

\( \mathbb{R} \) means the real numbers, \( \mathbb{C} \) is the complex numbers, and \( \mathbb{Q} \) is the rational numbers. If \( E \subset X \) is a set then \( \chi_E : X \to \mathbb{R} \) means

\[
\chi_E(x) := \begin{cases} 
1 & \text{if } x \in E; \\
0 & \text{if } x \in X \setminus E.
\end{cases}
\]

**Complex Analysis.**

1. Let \( f \) be analytic on \( \mathbb{C} \setminus \{0\} \) and suppose that, for all \( z \in \mathbb{C} \setminus \{0\} \),

\[
|f(z)| \leq |z|^{-1/3} + |z|^{3/4}.
\]

Show that \( f \) is constant.

2. Use residues or a substitution to show that

\[
\int_{-\infty}^{\infty} \frac{\log |x|}{1 + x^2} \, dx = 0
\]

and then use that fact plus residues to show

\[
\int_{-\infty}^{\infty} \frac{(\log |x|)^2}{1 + x^2} \, dx = \frac{\pi^3}{4}.
\]

3. Let \( f, g : \mathbb{C} \to \mathbb{C} \) be entire and suppose that

\[
|f(z)| \leq |g(z) + f(z)|
\]

for all \( z \in \mathbb{C} \). Show that \( \{f, g\} \) is a linearly dependent set: \( \exists \lambda_1, \lambda_2 \in \mathbb{C} \), not both equal to 0, such that \( \lambda_1 f(z) + \lambda_2 g(z) = 0 \) for all \( z \in \mathbb{C} \). (Hint: Divide)

4. Let \( \Omega := \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)) \). Find an analytic bijection \( f : \Omega \to \{z : \Re z > 0\} \). Express your \( f \) as a sequence of compositions, sketching the intermediate domains.

5. Let \( f : \mathbb{C} \to \mathbb{C} \) be entire and suppose that, for each \( z \in \mathbb{C} \), there is a \( k = k(z) \in \{1, 2, 3, \ldots\} \) such that \( f^{(k)}(z) = 0 \) (where \( f^{(k)} \) means \( f \)'s \( k \)th derivative). Show that \( f \) is a polynomial. (Hint: For each \( k \), set \( E_k := \{z \in \mathbb{C} : f^{(k)}(z) = 0\} \). What is the cardinality of \( E_k \)?)

6. Let \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) be analytic and suppose that, for every \( k \in \{1, 2, 3, \ldots\} \), there is a \( z_k \in \mathbb{C} \setminus \{0\} \) such that \( |z_k| < 1/2 \) and \( |z_k^k f(z_k)| > 1/2 \). Identify the type of singularity at \( z = 0 \) and show that there is a sequence \( \{\zeta_j\} \subset \mathbb{C} \setminus \{0\} \) such that \( \zeta_j \to 0 \) and \( f(\zeta_j) \to 1 \).
7. Define
\[ f(z) := \frac{z + 1}{z^2 + z - 2}. \]
Find a Laurent expansion for \( f \), of the form
\[ \sum_{n=-\infty}^{\infty} c_n (z+1)^n, \]
which converges to \( f \) in \( \{ z \in \mathbb{C} : 1 < |z+1| < 2 \} \).

**Real Analysis.**

8. Let \((M,d)\) be a metric space. Show that, if \( A, B \subset M \) are closed and \( A \cap B = \emptyset \), then there exist open sets \( U, V \) such that \( A \subset U, B \subset V \), and \( U \cap V = \emptyset \). (Hint: \( U \) and \( V \) are unions of open balls. How do you choose the radius of each ball?)

9. Let \((X, \mathcal{M}, \mu)\) be a measure space and let \( \phi : X \to [0, \infty] \) be measurable. For \( E \in \mathcal{M} \) define \( \lambda(E) := \int_E \phi \, d\mu = \int \phi \chi_E \, d\mu \). Use standard limit theorems to show that \( \lambda \) defines a measure on \( \mathcal{M} \) and that, if \( f : X \to [0, \infty] \) is measurable, then
\[ \int f \, d\lambda = \int f \, \phi \, d\mu. \]

You may use without proof the fact that there exists a sequence \( \{ \psi_n \}_{n=1}^{\infty} \) of non-negative measurable simple functions such that \( \psi_n (x) \leq \psi_{n+1} (x) \) for all \( x \) and \( n \), and \( \psi_n (x) \to f(x) \) pointwise as \( n \to \infty \).

10. Use standard limit theorems from measure theory and facts from calculus (about the integrals of exponentials, etc.) to show that
\[ \int_0^{\infty} \frac{xe^{-x}}{1 - e^{-x}} \, dx = \sum_{k=1}^{\infty} \frac{1}{k^2}, \]
where ‘\( dx \)’ means integration with respect to Lebesgue measure. (Hint: Use a geometric series.)

11. Enumerate the rationals \( \mathbb{Q} := \{ q_1, q_2, q_3, \ldots \} \), and define
\[ f(x) := \sum_{k=1}^{\infty} \frac{2^k}{k^2} \chi(q_k - 2^{-k}, q_{k+1} + 2^{-k})(x). \]

Show that
\[ \int_{\mathbb{R}} f(x) \, dx < \infty \]
(where ‘\( dx \)’ means integration with respect to Lebesgue measure), but that, for every \( p > 1 \) and every non-empty open \( U \subset \mathbb{R} \),
\[ \int_U f(x)^p \, dx = \infty. \]
12. Let $f : [a, b] \to \mathbb{R}$ be continuous, with $f(a) \leq f(b)$, and suppose that $f$ has no local maximum or minimum on $(a, b)$. Show that $f$ is non-decreasing on all of $[a, b]$: $\forall x, y \in [a, b], x < y \Rightarrow f(x) \leq f(y)$. (Hint: First show that, if $x \in [a, b]$, then $f(a) \leq f(x) \leq f(b)$.)

13. Let $f : [a, b] \to \mathbb{R}$ be continuous, with $[a, b] \subset \mathbb{R}$ and $a < b$. Use standard facts about continuous functions and the Riemann integral to show that

$$
\lim_{n \to \infty} \left(\int_{a}^{b} |f(x)|^n \, dx\right)^{1/n}
$$

exists and equals $\max_{[a, b]} |f(x)|$.

14. Show that, if $\{a_k\}_1^\infty$ is any sequence of non-negative numbers and $0 < p < q < \infty$ then

$$
\left(\sum_{k=1}^{\infty} a_k^q\right)^{1/q} \leq \left(\sum_{k=1}^{\infty} a_k^p\right)^{1/p}.
$$