Algebra Qualifying Exam — January 2021

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Format

The exam contains three sections (Sections A, B, and C).

Each section contains three numbered problems (Problems 1, 2, and 3).

Finally, each numbered problem has a certain number of lettered subproblems (Parts (a), (b), (c), etc.).

Instructions

You have three hours to complete this exam.

When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass:

Four numbered problems solved completely.

The set of problems solved completely must include one from each of sections A,B, and C.

Substantial progress on two other problems.

MS Pass:

Nine lettered subproblems solved completely.

The set of subproblems solved must include two subsets of three subproblems that are all from the same section.

The set of subproblems solved must include one from each sections A, B, and C.

Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order $351 = 3^3 \cdot 13$.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each prime p dividing the order of the group G.

   (b) Show that G is not simple.

   (c) Show that G is solvable. For this question you may not use the Feit-Thompson Theorem or Burnside's Theorem.

2. Fix p a prime and let G be the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

   where $a, b, c \in \mathbb{F}_p$, and where the operation is multiplication.

   (a) Prove that the subgroup H of matrices where $a = c = 0$ is normal.

   (b) Express the group G/H as a direct product of cyclic groups. (Your answer can be a single cyclic group if G/H is cyclic.) You must justify your answer.

   (c) Prove that for each prime p, there is a group of order p^3 that is not abelian. (You might remember that all groups of order p and p^2 are abelian, when p is prime. It ends there!)

3. Let P be a finite p-group for p a prime, and suppose that P acts on a finite set S.

   (a) Denote by S^P the set of elements of S that are fixed by every element of P:

$$S^P = \{s \in S : g \cdot s = s \text{ for all } g \in P\}.$$

   Prove that

$$\#S \equiv \#S^P \pmod{p},$$

   where # denotes the cardinality of the set following it.

   (b) Suppose now that P acts transitively on S. Prove that $\#S$ is a power of p.

Section B

4. Consider the ring
$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}, i^2 = -1\},$$
with the usual addition and multiplication.

   (a) Show that $\mathbb{Z}[i]$ is in fact ta Euclidean domain with norm $N(a+bi) = a^2 + b^2$.

   (b) Let $I = (\alpha) \subseteq \mathbb{Z}[i]$ be a principal ideal with $\alpha \neq 0$. Show that every coset of $\mathbb{Z}[i]/I$ contains a representative $a + bi$ with $N(a + bi) < N(\alpha)$.

   (c) Let I be an arbitrary nonzero ideal in $\mathbb{Z}[i]$. Show that the quotient ring $\mathbb{Z}[i]/I$ is finite.

5. Let R be a commutative ring with 1 and M be a (left) R-module.

   (a) If N is also a left-R-module and $\varphi\colon M \to N$ is an R-module homomorphism, prove that
$$\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$$
where Tor is the subset of torsion elements.

   (b) Suppose further that R is an integral domain. Show that $\mathrm{Tor}(M)$ is a sub-R-module of M.

   (c) Give an example of a ring R and a module M such that $\mathrm{Tor}(M)$ is not a sub-R-module of M. Justify your answer.

6. (a) Give one representative for each similarity class of 5×5 matrices A with entries in $\mathbb{Q}$ that satisfy $A^3 = I$ but $A \neq I$. Justify your answer.

   (b) Of the similarity classes you have found in part 6(a), do any of them contain a diagonal matrix? Justify your answer. (Here we mean that the matrices in the similarity class can be diagonalized over $\mathbb{Q}$.)

   (c) Give one representative for each similarity class of 5×5 matrices A with entries in $\mathbb{F}_3$ that satisfy $A^3 = I$ but $A \neq I$. Justify your answer.

7. Let k be a field. Consider the ring homomorphism $\varphi\colon k[x, y, z] \to k[t]$ given by $\varphi(x) = t, \varphi(y) = t^2$ and $\varphi(z) = t^3$, and let I denote the kernel of φ.

   (a) Show that I is generated by $y - x^2$ and $z - x^3$. Furthermore explain why I is a prime ideal.

   (b) Prove that $\{y - x^2, z - x^3\}$ is a Gröbner basis for I in some term ordering.

   (c) Verify that $z - xy \in I$ by writing $z - xy$ as an explicit $k[x, y, z]$-linear combination of $y - x^2$ and $z - x^3$.

Section C

8. In this problem, let ζ be a primitive eighth root of unity.

   (a) Give the lattice of all fields containing $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta)$.

   (b) For each of the fields you enumerated in part (a), give a generator in terms of ζ. (For example, you could say that a field is generated by $\zeta^3 + 4$.)

   (c) For each of the fields you enumerated in part (a), give a generator in terms of radicals of rational numbers. (For example, you could say that a field is generated by $\sqrt{17}$.)

9. (a) Let f be an irreducible polynomial of degree n over $\mathbb{Q}$ with splitting field K. Prove that if $\mathrm{Gal}(K/\mathbb{Q})$ is abelian, then $[K : \mathbb{Q}] = n$.

   (b) Let p be a prime and let f be an irreducible polynomial of degree p over $\mathbb{Q}$ with splitting field K. Prove that $\#\,\mathrm{Gal}(K/\mathbb{Q}) = pm$ for m relatively prime to p, and that if $\mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup of order m, then $m = 1$.

10. (a) How many irreducible polynomials of degree 3 are there in $\mathbb{F}_2[x]$?

    (b) Give a complete list of irreducible polynomials of degree 3 in $\mathbb{F}_2[x]$.

    (c) Let K be a field with 8 elements. How many elements $\alpha \in K$ generate the multiplicative group $K^\times$?

    (d) How many primitive elements are there for the extension $K/\mathbb{F}_2$? (In other words, how many β are there in $\overline{\mathbb{F}}_2$ such that $K = \mathbb{F}_2(\beta)$?)