Algebra Qualifying Exam — August 2020

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Format

The exam contains three sections (Sections A, B, and C).

Each section contains three numbered problems (Problems 1, 2, and 3).

Finally, each numbered problem has a certain number of lettered subproblems (Parts (a), (b), (c), etc.).

Instructions

You have three hours to complete this exam.

When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass:

Four numbered problems solved completely.

The set of problems solved completely must include one from each of sections A,B, and C.

Substantial progress on two other problems.

MS Pass:

Nine lettered subproblems solved completely.

The set of subproblems solved must include two subsets of three subproblems that are all from the same section.

The set of subproblems solved must include one from each sections A, B, and C.

Section A

1. (a) Show that S_3 acts transitively on 6 elements by giving an explicit example.

   (b) Any transitive action of S_3 on a set with 6 elements gives an injective group homomorphism $S_3 \hookrightarrow S_6$. For the action you have given in part 1(a), give this homomorphism explicitly.

   (c) Consider the "usual" injective group homomorphism $S_3 \hookrightarrow S_6$ given by sending $(12) \mapsto (12)$ and $(123) \mapsto (123)$. If H_1 is the image of S_3 in S_6 under the homomorphism of part 1(b), and H_2 is the image of S_3 in S_6 under the "usual" injective homomorphism, are H_1 and H_2 conjugate in S_6? Briefly justify your answer.

2. In this problem, G is a finite group.

   (a) Show that if $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G, then G is abelian.

   (b) Let p be a prime and P be a p-group. Show that $Z(P)$ is nontrivial.

   (c) Show that if P has order p^2 then P is abelian.

   (d) Show that every p-group is solvable.

3. Let G be a group of order 63.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for all primes p dividing 63.

   (b) Show that if the Sylow 3-subgroup of G is normal, then G is abelian.

   (c) Let H be a group of order 9. Show that there is only one nontrivial action of the group H on the group C_7 (up to automorphisms of H).

   (d) Show that there are exactly four isomorphism classes of groups of order 63.

Section B

In this section if I is an ideal in a commutative ring R the radical of I will be denoted $\sqrt{I}$ and it is the set

$$\{a \in R : \exists n \in \mathbb{Z}_{\geq 0} \quad a^n \in I\}.$$

4. In this problem R is a commutative ring with 1, and I and J are ideals in R.
   (a) Show that if $I + J = R$, then $IJ = I \cap J$.
   (b) Show that if $I + J = R$ then $R/(I \cap J) = R/I \oplus R/J$.
   (c) Conclude that for distinct primes $p_1, p_2, \ldots, p_r \in \mathbb{Z}$ and positive integers $n_1, n_2, \ldots, n_r$, we have
   $$\mathbb{Z}/(p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}) \cong \mathbb{Z}/(p_1^{n_1}) \oplus \mathbb{Z}/(p_2^{n_2}) \oplus \cdots \oplus \mathbb{Z}/(p_r^{n_r}).$$

5. In this problem A and B are commutative rings with 1.
   (a) Show that $\sqrt{(0)}$ is the ideal of nilpotent elements in A.
   (b) Show that $\sqrt{IJ} = \sqrt{I \cap J}$ for every pair of ideals I, J of A.
   (c) Let $\sigma \colon A \to B$ be a ring homomorphism. Show that the inverse image of a prime ideal is prime. Show that this is not necessarily the case for maximal ideals (i.e., exhibit an explicit ring homomorphism and a maximal ideal in the codomain whose inverse image is not maximal).
   (d) Show that the map $A \to A/\sqrt{(0)}$ induces a bijection between the prime ideals of A and the prime ideals of $A/\sqrt{(0)}$.

6. (a) How many similarity classes of 6×6 matrices A with entries in $\mathbb{F}_2$ are there that satisfy $A^6 = I$ but $A^n \neq I$ for every $n \in \{1, \ldots, 5\}$? Justify your answer; you do not need to give a representative for each class.
   (b) Of the similarity classes you have found in part 6(a), give a representative of a class of matrices that is diagonalizable over $\mathbb{F}_2$ or state that one doesn't exist. Similarly, give a representative of a class of matrices that is not diagonalizable over $\mathbb{F}_2$ or state that one doesn't exist.
   (c) How many similarity classes of 6×6 matrices A with entries in $\mathbb{F}_3$ are there that satisfy $A^6 = I$ but $A^n \neq I$ for every $n \in \{1, \ldots, 5\}$? Justify your answer; you do not need to give a representative for each class.

7. (a) Let R be a commutative ring with 1, let M be an R-module, and let I be an ideal in R. Show that
   $$M \otimes_R (R/I) \cong M/IM$$
   as R-modules, where $IM = \{am : a \in I, m \in M\}$.
   (b) Prove that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$.
   (c) Let p be a prime. Let $R = \mathbb{Z}/(p^2)$ and $M = \mathbb{Z}/(p)$. View M as an R-module via the reduction modulo p map. Show that M is not a flat R-module.

Section C

8. (a) Show that $K = \mathbb{Q}(\sqrt{2+\sqrt{3}})$ is Galois over $\mathbb{Q}$ with Galois group $C_2 \times C_2$.

   (b) It is a fact that if K is an extension of $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong C_2 \times C_2$, then there are $x, y \in \mathbb{Q}$, nonsquares and such that xy is not a square, such that $K = \mathbb{Q}(\sqrt{x}, \sqrt{y})$. (You do not need to prove this fact.)

   Give $x, y \in \mathbb{Q}$ such that $\mathbb{Q}(\sqrt{2+\sqrt{3}}) = \mathbb{Q}(\sqrt{x}, \sqrt{y})$.

9. In this problem, let $p > 2$ be a prime and for all n let $\Phi_n(x) \in \mathbb{Z}[x]$ be the nth cyclotomic polynomial.

   (a) Show that $\Phi_p(x+1)$ is irreducible over $\mathbb{Z}$.

   (b) Conclude that $\Phi_p(x)$ is irreducible over $\mathbb{Q}$.

   (c) Prove that $\Phi_{2p}(x) = \Phi_p(-x)$ for all odd primes p.

10. (a) How many distinct roots does the polynomial $x^3 - 1$ have in $\overline{\mathbb{F}}_3$?

   (b) How many distinct roots does the polynomial $x^7 - 1$ have in $\overline{\mathbb{F}}_3$?

   (c) Let K be the splitting field of $x^7 - 1$ over $\mathbb{F}_3$. What is the degree of K over $\mathbb{F}_3$?

   (d) Draw the lattice of all subfields of K. (You need not give generators for these subfields.)

   (e) How many elements $\alpha \in K$ generate the multiplicative group $K^\times$?

   (f) How many primitive elements are there for the extension $K/\mathbb{F}_3$? (In other words, how many β are there such that $K = \mathbb{F}_3(\beta)$?)