Analysis Qualifying Exam
August, 2021

Passing levels:

MS: You must do, in total, at least 4 problems completely correctly, or 3 completely correctly with substantial progress on 2 others. You are free to choose the problems from either or both sections.

PhD: You must do: a) at least 2 completely correctly from **each** of the two sections; and b) at least 6 completely correctly in total, or 5 completely correctly with substantial progress on 2 others.

Notation: $\mathbf{R}$ means the real numbers and $\mathbf{C}$ means the complex numbers. If $z \in \mathbf{C}$ then $\Re z$ means z's real part and $\Im z$ is its imaginary part. If $f : X \to Y$ and $A \subset X$ then $f[A] := \{f(x) : x \in A\}$.

Section I: Real analysis.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(X, \mathcal{M}, \mu)$. Show that, for all $\epsilon > 0$, there is a $\delta > 0$ so that, if $E \in \mathcal{M}$ and $\mu(E) < \delta$, then $\int_E |f| \, d\mu < \epsilon$.

2. Let $\phi \in L^\infty(\mathbf{R}, \mathcal{L}, m)$ (the usual Lebesgue space on the line). For $t > 0$ define

$$G(t) := \int_{\mathbf{R}} e^{-t|x|} \, \phi(x) \, dx,$$

where the integral is assumed to be a Lebesgue integral. Show that G is defined and continuous on all of $(0, \infty)$. You may assume standard calculus facts about the exponential function.

3. Let (X, d) be a metric space. Show that, if $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X, then

$$\lim_{n\to\infty} d(x_n, y_n)$$

exists as a real number, where the limit is taken with respect to the usual $|\cdot|$-based metric.

4. Let (X, d) be a connected metric space, and let $f : X \to \mathbf{R}$ have the property that, for every $p \in X$, there is an $r > 0$ such that f is constant on $B(p; r) := \{x : d(x, p) < r\}$. Show that f is constant on all of X.

5. Find, with justification via appropriate limit theorems,

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-3x} \, dx.$$

You may assume standard calculus facts about the exponential function.

6. Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable everywhere, and suppose that

$$\lim_{x\to\infty} f'(x) = 0.$$

Show that

$$\lim_{x\to\infty} \frac{f(x)}{x}$$

exists and equals 0.

7. Consider $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(x, y) := (x^2 + 2xy, xy + y^2)$. Show that, if $(a, b) \neq (0, 0)$, there is an open $U \subset \mathbf{R}^2$ with $(a, b) \in U$ such that f is one-to-one on U, $f[U]$ is open, and there is a differentiable $g : f[U] \to U$ such that $g(f(x, y)) = (x, y)$ for all $(x, y) \in U$.

8. Let (X, d) be a compact metric space, where "compact" means "every open cover of X has a finite subcover". Show that every infinite sequence $\{x_n\} \subset X$ has a subsequence converging to some $p \in X$.

Section II: Complex analysis.

1. Find an analytic bijection $f : \{z : \Re z > 0, |z| > 1\} \to \{z : \Re z > 0\}$. Write your bijection as a sequence of compositions of analytic bijections, with sketches of the intermediate domains.

2. Use residues to show that

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(1 + x^2)^2}\, dx = \frac{3\pi}{2e^2}.$$

3. Let $\overline{D} := \{z : |z| \leq 1\} \subset U$ for some open $U \subset \mathbf{C}$. Suppose that $f : U \to \mathbf{C}$ is analytic and $|f(z)| < 1$ for all z with $|z| = 1$. Show that there is a unique ζ with $|\zeta| < 1$ such that $f(\zeta) = \zeta$.

4. State a form of the maximum principle for analytic functions and use it to prove the following: *Let $U \subset \mathbf{C}$ be a connected open set, with $f : U \to \mathbf{C}$ analytic. Write $f := u + iv$, where u and v are f's real and imaginary parts. Suppose there is some $a \in U$ such that, for all $z \in U$,*

$$3u(a) + 2v(a) \geq 3u(z) + 2v(z).$$

Conclusion: f is constant.

5. Suppose that f is analytic and zero-free on $\{z : 0 < |z| < 1\}$. Show that, if 0 is not a removable singularity for f or $1/f$, then 0 is an essential singularity for both of them.

6. Find the complex numbers $\{c_n\}_{-\infty}^{\infty}$ such that

$$\sum_{-\infty}^{\infty} c_n (z - 3)^n$$

converges to

$$f(z) := \frac{4z + 1}{z^2 - 3z - 10}$$

on the annulus $\{z: \ 2 < |z-3| < 5\}$.

7. Let $f : \mathbf{C} \to \mathbf{C}$ be entire, and suppose there are positive numbers a and b such that $|f(z)| > a$ whenever $|z| > b$. Show that f is a polynomial.

8. Let $p(z)$ and $q(z)$ be two non-trivial polynomials with different degrees. Show that there is no entire $f : \mathbf{C} \to \mathbf{C}$ such that

$$|p(z)| \leq |f(z)| \leq |q(z)|$$

for all $z \in \mathbf{C}$.