ANALYSIS QUALIFYING EXAM
24 August 2020

You have 3 hours to do the exam.

The exam has two sections, Complex and Real. Passing at the PhD level is accomplished by solving at least two problems in each section while either solving a total of 6 problems or solving 5 problems and making significant progress on two others.

Do not mark this exam in any way which would identify you. The proctor will assign you a letter. Write this letter at the upper right of each of your work sheets.

$\mathbf{R}$ means the real numbers, $(\mathbf{R}, \mathcal{L}, m)$ means the Lebesgue measure space on $\mathbf{R}$, and $\mathbf{C}$ is the complex numbers. If $z \in \mathbf{C}$ then $\Re z$ is z's real part and $\Im z$ is z's imaginary part.

Complex Analysis.

1. Let f be entire and suppose that $|f(z)| > 1$ whenever $|z| > 1$. Show that f is a polynomial.

2. Let f be entire with $f(0) = 0$, and let $\{a_n\}_1^\infty \subset \mathbf{C}$ be a sequence of complex numbers such that $\sum_1^\infty |a_n| < \infty$. Show that

$$\phi(z) := \sum_1^\infty f(a_n z)$$

converges for all $z \in \mathbf{C}$ and defines an entire function.

3. Let $\Omega := \mathbf{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Find an analytic bijection $f : \Omega \to \{z : \Re z > 0\}$. Express your f as a sequence of compositions, sketching the intermediate domains.

4. Define $D := \{z \in \mathbf{C} : |z| < 1\}$ and $\overline{D} := \{z \in \mathbf{C} : |z| \leq 1\}$. Let $f : D \to \overline{D}$ be analytic and suppose that $f(1/2) = f(-1/2) = 0$. Show that, for all $z \in D$,

$$|f(z)| \leq \left| \frac{z^2 - 1/4}{1 - (z/2)^2} \right|.$$

5. Define $D := \{z \in \mathbf{C} : |z| < 1\}$. Suppose that f and $1/f$ are both analytic on $D \setminus \{0\}$ and 0 is not a removable singularity for f or $1/f$. Show that 0 is an essential singularity for both functions.

6. Use residues to show that, for any $a \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+1)(x^2+4)}\, dx = \pi \left(\frac{e^{-a}}{3} - \frac{e^{-2a}}{6} \right).$$

7. Find a Laurent series for

$$f(z) := \frac{5z+1}{z^2+2z-15}$$

converging in the annulus $\{z \in \mathbf{C}: \ 2 < |z-1| < 6\}$. Your series will have the form

$$\sum_{n=-\infty}^{\infty} c_n (z-1)^n.$$

Real Analysis.

1. Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) := (x^4 - y^3, x^2 + y)$ and set $H := \{(x,y) \in \mathbf{R}^2 : \ x > 0\}$. Show that, for every $(a,b) \in H$, there is an open set U with $(a,b) \in U \subset H$ on which f is one-to-one, and there is a differentiable $g : f[U] \to U$ such that $g(f(x,y)) = (x,y)$ for all $(x,y) \in U$.

2. For $f \in L^1(\mathbf{R}, \mathcal{L}, m)$ and $t \in \mathbf{R}$ define

$$\widehat{f}(t) := \int_{\mathbf{R}} f(x) \exp(-2\pi i x t) \, dm(x).$$

Show that $\widehat{f}(t)$ is a continuous function of t.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. A family $\mathcal{F}$ of measurable functions is said to be uniformly integrable if, for every $\epsilon > 0$, there is a $\delta > 0$ so that, for all $E \in \mathcal{M}$ and $f \in \mathcal{F}$, $\mu(E) < \delta \Rightarrow \int_E |f| \, d\mu < \epsilon$. Show that any finite $\mathcal{F} \subset L^1(X, \mathcal{M}, \mu)$ is uniformly integrable. (You may ignore the trivial case in which $\mathcal{F} = \emptyset$.)

4. Let $f : [0,1] \to \mathbf{R}$ be continuous, $f(0) = 0$, and with f' existing and finite on all of $(0,1)$. Show that, if f' is increasing on $(0,1)$, so is $g(x) := f(x)/x$.

5. Let $f : (0,1) \to \mathbf{R}$ be uniformly continuous on $(0,1)$ (assuming the usual $|\cdot|$ metric). Show that

$$\lim_{x \to 0^+} f(x)$$

and

$$\lim_{x \to 1^-} f(x)$$

both exist.

6. Suppose $\{a_k\}_1^\infty$ is a sequence of complex numbers such that

$$\sum_1^\infty a_k b_k$$

converges for every complex sequence $\{b_k\}_1^\infty$ having the property that

$$\lim_{k \to \infty} b_k = 0.$$

Show that

$$\sum_1^\infty |a_k| < \infty.$$

7. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous on all of $\mathbf{R}$ (with the usual $|\cdot|$ metric) and

$$\lim_{|x| \to \infty} f(x) = 0.$$

Show that f is uniformly continuous on $\mathbf{R}$.