ANALYSIS QUALIFYING EXAM

18 May 2020

Passing at the PhD level is accomplished by solving at least two problems in each section while either solving a total of 6 problems or solving 5 problems and making significant progress on two others.

R means the real numbers, **C** is the complex numbers, and **Q** is the rational numbers. If $E \subset X$ is a set then $\chi_E : X \to \mathbf{R}$ means

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \in X \setminus E. \end{cases}$$

Complex Analysis.

1. Let f be analytic on $\mathbf{C} \setminus \{0\}$ and suppose that, for all $z \in \mathbf{C} \setminus \{0\}$,

$$|f(z)| \leq |z|^{-1/3} + |z|^{3/4}.$$

Show that f is constant.

2. Use residues or a substitution to show that

$$\int_{-\infty}^{\infty} \frac{\log |x|}{1+x^2}\, dx = 0$$

and then use that fact plus residues to show

$$\int_{-\infty}^{\infty} \frac{(\log |x|)^2}{1+x^2}\, dx = \frac{\pi^3}{4}.$$

3. Let $f, g : \mathbf{C} \to \mathbf{C}$ be entire and suppose that

$$|f(z)| \leq |g(z) + f(z)|$$

for all $z \in \mathbf{C}$. Show that $\{f, g\}$ is a linearly dependent set: $\exists \lambda_1, \lambda_2 \in \mathbf{C}$, not both equal to 0, such that $\lambda_1 f(z) + \lambda_2 g(z) = 0$ for all $z \in \mathbf{C}$. (Hint: Divide)

4. Let $\Omega := \mathbf{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Find an analytic bijection $f : \Omega \to \{z : \Re z > 0\}$. Express your f as a sequence of compositions, sketching the intermediate domains.

5. Let $f : \mathbf{C} \to \mathbf{C}$ be entire and suppose that, for each $z \in \mathbf{C}$, there is a $k = k(z) \in \{1, 2, 3, \ldots\}$ such that $f^{(k)}(z) = 0$ (where $f^{(k)}$ means f's k^{th} derivative). Show that f is a polynomial. (Hint: For each k, set $E_k := \{z \in \mathbf{C} : f^{(k)}(z) = 0\}$. What is the cardinality of E_k?)

6. Let $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ be analytic and suppose that, for every $k \in \{1, 2, 3, \ldots\}$, there is a $z_k \in \mathbf{C} \setminus \{0\}$ such that $|z_k| < 1/2$ and $|z_k^k f(z_k)| > 1/2$. Identify the type of singularity at $z = 0$ and show that there is a sequence $\{\zeta_j\} \subset \mathbf{C} \setminus \{0\}$ such that $\zeta_j \to 0$ and $f(\zeta_j) \to 1$.

7. Define

$$f(z) := \frac{z+1}{z^2+z-2}.$$

Find a Laurent expansion for f, of the form

$$\sum_{-\infty}^{\infty} c_n (z+1)^n,$$

which converges to f in $\{z \in \mathbf{C} : \ 1 < |z+1| < 2\}$.

Real Analysis.

8. Let (M,d) be a metric space. Show that, if $A, B \subset M$ are closed and $A \cap B = \emptyset$, then there exist open sets U, V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. (Hint: U and V are unions of open balls. How do you choose the radius of each ball?)

9. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\phi : X \to [0, \infty]$ be measurable. For $E \in \mathcal{M}$ define $\lambda(E) := \int_E \phi \, d\mu = \int \phi \, \chi_E \, d\mu$. Use standard limit theorems to show that λ defines a measure on $\mathcal{M}$ and that, if $f : X \to [0, \infty]$ is measurable, then

$$\int f \, d\lambda = \int f \, \phi \, d\mu.$$

You may use without proof the fact that there exists a sequence $\{\psi_n\}_1^\infty$ of non-negative measurable simple functions such that $\psi_n(x) \leq \psi_{n+1}(x)$ for all x and n, and $\psi_n(x) \to f(x)$ pointwise as $n \to \infty$.

10. Use standard limit theorems from measure theory and facts from calculus (about the integrals of exponentials, etc.) to show that

$$\int_0^\infty \frac{x\, e^{-x}}{1 - e^{-x}} \, dx = \sum_1^\infty \frac{1}{k^2},$$

where 'dx' means integration with respect to Lebesgue measure. (Hint: Use a geometric series.)

11. Enumerate the rationals $\mathbf{Q} := \{q_1, q_2, q_3, \ldots\}$, and define

$$f(x) := \sum_1^\infty \frac{2^k}{k^2} \chi_{(q_k - 2^{-k}, q_k + 2^{-k})}(x).$$

Show that

$$\int_{\mathbf{R}} f(x) \, dx < \infty$$

(where 'dx' means integration with respect to Lebesgue measure), but that, for every $p > 1$ and every non-empty open $U \subset \mathbf{R}$,

$$\int_U f(x)^p \, dx = \infty.$$

12. Let $f : [a,b] \to \mathbf{R}$ be continuous, with $f(a) \le f(b)$, and suppose that f has no local maximum or minimum on (a,b). Show that f is non-decreasing on all of $[a,b]$: $\forall x, y \in [a,b] (x < y \Rightarrow f(x) \le f(y))$. (Hint: First show that, if $x \in [a,b]$, then $f(a) \le f(x) \le f(b)$.)

13. Let $f : [a,b] \to \mathbf{R}$ be continuous, with $[a,b] \subset \mathbf{R}$ and $a < b$. Use standard facts about continuous functions and the Riemann integral to show that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n \, dx \right)^{1/n}$$

exists and equals $\max_{[a,b]} |f(x)|$.

14. Show that, if $\{a_k\}_1^\infty$ is any sequence of non-negative numbers and $0 < p < q < \infty$ then

$$\left(\sum_1^\infty a_k^q \right)^{1/q} \le \left(\sum_1^\infty a_k^p \right)^{1/p}.$$