Analysis Qualifying Exam
January, 2021

Passing levels:

MS: You must do, in total, at least 4 problems completely correctly, or 3 completely correctly with substantial progress on 2 others. You are free to choose the problems from either or both sections.

PhD: You must do: a) at least 3 completely correctly from each of the two sections; and b) at least 8 completely correctly in total, or 7 completely correctly with substantial progress on 2 others.

Section I: Real analysis.

1. Let $f : [0,1] \to \mathbf{R}$ be continuous (in the usual $|\cdot|$-based sense) and one-to-one, and suppose that $f(0) < f(1)$. Show that $\forall x, y \in [0,1](x < y \Rightarrow f(x) < f(y))$.

2. Let $f \in L^1(\mathbf{R}, \mathcal{L}, m)$, where $(\mathbf{R}, \mathcal{L}, m)$ is the Lebesgue measure space. Show that

$$\lim_{n\to\infty} \int f(x)\, \sin^2(nx)\, dx$$

exists and equals

$$\frac{1}{2} \int f(x)\, dx.$$

Hint: First show the result for $f = \chi_{(a,b)}$, where (a,b) is a bounded interval.

3. Let A, B be non-empty subsets of $\mathbf{R}^d$, equipped with the usual $\|\cdot\|$-based metric. Show that, if A is compact and B is closed, then there exist points $a \in A$ and $b \in B$ such that, for all $x \in A$ and $y \in B$,

$$\|a - b\| \le \|x - y\|.$$

Give an example in $\mathbf{R}$ to show that such points a and b need not exist if A and B are merely assumed to be closed.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous everywhere (in the usual sense) and suppose that

$$\lim_{|t|\to\infty} f(t)$$

exists as a real number. Show that f is uniformly continuous on $\mathbf{R}$.

5. Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) := (x^2 - y^2, x(y+1))$. Show that, if $(a,b) \in \mathbf{R}^2$ and $|a| > 1/2$, there is an open set $U_{(a,b)}$ containing (a,b) such that f is one-to-one on $U_{(a,b)}$, $f[U_{(a,b)}]$ is open, and f has a differentiable inverse function $f^{-1} : f[U_{(a,b)}] \to U_{(a,b)}$.

6. Find the limit

$$\lim_{n\to\infty} \int_0^n e^{-2x} \left(\sum_0^n (-1)^k \frac{x^{2k}}{(2k)!} \right) dx$$

(where $(-1)^0 := 1$), justifying your answer using appropriate limit theorems from measure theory and basic calculus facts about exponentials and power series.

7. Let $f : [0,1] \to \mathbf{C}$ be continuous in the usual sense, and define

$$M := \max_{x\in[0,1]} |f(x)|,$$

which we know exists because $[0,1]$ is compact. Show that

$$\lim_{n\to\infty} \left(\int_0^1 |f(x)|^n \, dx \right)^{1/n}$$

exists and equals M.

Section II: Complex analysis.

1. Use residues to show that

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi x)}{e^{\pi x} + e^{-\pi x}} \, dx = \frac{1}{e^{\pi} + e^{-\pi}}.$$

2. Let $f : \mathbf{C} \to \mathbf{C}$ be entire and not a polynomial. Show that, for all $a \in \mathbf{C}$ and $0 < r < \infty$,

$$f\left[\mathbf{C} \setminus \Delta(a;r)\right]$$

is dense in $\mathbf{C}$, where $\Delta(a;r) := \{z \in \mathbf{C} : \ |a-z| < r\}$.

3. Set $D := \{z \in \mathbf{C} : \ |z| < 1\}$. Write down, as a sequence of compositions of analytic bijections, an analytic bijection $f : \Omega \to \mathcal{H}$, where

$$\Omega := D \setminus ((-1, -1/2] \cup [1/2, 1))$$
$$\mathcal{H} := \{z \in \mathbf{C} : \ \Re z > 0\}.$$

4. Let $f : \mathbf{C} \to \mathbf{C}$ be entire, with $f(0) = 0$. Show that

$$\phi(z) := \sum_{1}^{\infty} f(z/n^2)$$

converges for all $z \in \mathbf{C}$ and defines an entire function.

5. Let $\Omega \subset \mathbf{C}$ be a connected open set, and $f : \Omega \to \mathbf{C}$ and $g : \Omega \to \mathbf{C}$ both analytic. Suppose that $\overline{f}g$ is analytic, where $\overline{f}$ means f's complex conjugate. Show that either f is constant or g is identically 0 on Ω.

6. Let

$$f(z) := \frac{3z - 7}{z^2 - 5z + 6}.$$

Find the Laurent expansion for f which is valid in $ann(1; 1, 2) := \{z \in \mathbf{C} : \; 1 < |z - 1| < 2\}$.

7. State some form of the Maximum Principle for functions f analytic on a connected open $\Omega \subset \mathbf{C}$. Use it (with other standard facts about analytic functions) to prove the following: If $\Omega \subset \mathbf{C}$ is connected and open, $f : \Omega \to \mathbf{C}$ is analytic, and there is some $a \in \Omega$ such that

$$\Re f(a) \leq \Re f(z)$$

for all $z \in \Omega$, then f is constant. We are using $\Re\alpha$ to mean the real part of a complex number α.