Numerical Analysis PhD Qualifying Exam

May 2020

Instructions:
Four problems must be completed, and one problem must be attempted. At least two problems from 1–4 (group 1) and at least two problems from 4–7 (group 2) must be completed. Note that Problem 4 can count towards either group, but not both. To have attempted a problem, you must correctly outline the main idea of the solution and begin the calculation, but need not finish it. You have three hours to complete the exam.

1. For the equation $f(x) = 0$, suppose x_* is a triple root, i.e. $f(x_*) = f'(x_*) = f''(x_*) = 0$ but $f'''(x_*) \neq 0$. If one uses Newton's method
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
to compute this root starting from an initial condition x_0 that is close to this root, will this method converge? If not, explain why. If yes, determine its convergence rate.

2. Compute the first three steps of the Jacobi and Gauss–Seidel iterations with starting vector $[1, 0, 1]^T$ by hand for the following linear system of equations:
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

3. Consider the following numerical formula to approximate the integral
$$\int_0^1 x f(x) dx \approx a_1 f(0) + a_2 f(\frac{1}{2}) + a_3 f(1),$$
where $f(x)$ is a smooth function. Determine the coefficients a_1, a_2 and a_3 so that this formula is accurate for polynomials $f(x)$ of as high a degree as possible.

4. Show that the midpoint method
$$Y_{n+1} - Y_n \;=\; h\, f\Big(\, x_n + (h/2),\; Y_n + (h/2) f(x_n, Y_n) \Big)$$
has the discretization error of order h^2.

   Note: You may need the formula for the Taylor series in two variables:
$$f(x + \Delta x,\, y + \Delta y) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\Delta x \, \frac{\partial}{\partial x} + \Delta y \, \frac{\partial}{\partial y} \right)^i f(x, y)\,.$$

5. (a) Propose a 2nd-order accurate discretization of the equation
$$(p(x) u_x)_x = q(x) u + r(x)\,. \qquad (1)$$

   (b) Use this discretization to set up a linear system for the boundary-value problem given by Eq. (1) and by the boundary conditions
$$u_x(0) = a, \qquad u(1) = b;$$
where a, b are some given constants. Use $h = 1/3$ and write out each equation in the linear system in question. Make sure to use the 2nd-order accurate approximation for the Neumann boundary conditions.

6. Consider Problem 5 above, where it is given that $p(x) > 0$ for $x \in [0, 1]$. Explain how you would solve this boundary-value problem by the shooting method (with an arbitrary h, of course). You should describe and explain all relevant details, but should *not* solve any equations.

7. Consider a unidirectional wave equation

$$u_t = c\,u_x, \qquad -\infty < x < \infty \tag{2}$$

where $c =$const.

Use the von Neumann analysis to determine if there is any range of parameter

$$\mu = \frac{c\tau}{h} > 0$$

for which the scheme

$$\frac{U_m^{n+1} - U_m^n}{\tau} = c\,\frac{U_m^n - U_{m-1}^n}{h}\,,$$

approximating (2), would be stable.