Differential Equations PhD Qualifying Exam
University of Vermont
January 11, 2017

Name: ____________________________

➤ Time allowed: 3 hours.
➤ Brains only: No calculators or other electronic gadgets allowed.
➤ **Two problems from each section must be completed correctly, and one additional problem from each section must be attempted**. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations, but do not need to finish them. **Numerical criteria for passing**: A problem is considered completed (attempted) if a grade for it is $\geq 85\%$ ($\geq 60\%$).

Section 1, ODE

1. Draw the phase portrait for the system

$$\dot{x} = x(2 - x - y)$$
$$\dot{y} = x - y$$

and identify the fixed points and their stability.

2. Solve the non-homogeneous linear system

$$\dot{\vec{x}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

with the initial condition $\vec{x}(0) = [1\ 0]^\top$.

3. Express the linear system of ODEs

$$\dot{x_1} = ax_1 - bx_2$$
$$\dot{x_2} = bx_1 + ax_2$$

in polar coordinates, where $r^2 = x_1^2 + x_2^2$ and $\theta = \tan^{-1}(x_2/x_1)$. The result should have a very simple form. Then solve using the initial conditions $r(0) = r_0,\ \theta(0) = \theta_0$.

4. Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in $(\mu,\ a)$ space at which Hopf bifurcations occur.

5. For $0 \le x \le \pi$, solve the problem

$$\phi_t = \phi_{xx} + w(x,t),$$

$$\phi(0,t) = 0, \quad \phi_x(\pi,t) = 0,$$

$$\phi(x,0) = f(x).$$

6. Solve the following 2D heat equation on a circular disk as simply as possible:

$$u_t = \nabla^2 u,$$

$$u(a,\theta,t) = 0,$$

$$u(r,\theta,0) = f(r).$$

Here a is the radius of the disk, and $f(r)$ is a prescribed arbitrary function.

7. Use the method of characteristics to solve the problem

$$\rho_t - x\rho_x = \rho + t, \qquad -\infty < x < \infty,$$

$$\rho(x,0) = f(x),$$

and express your solution explicitly in terms of the function $f(x)$.

8. Consider the following eigenvalue problem,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda - \frac{1}{r^2}\right)\phi = 0, \qquad 0 < r \le 3,$$

$$\phi(0) \text{ is finite}; \quad \phi(3) = 0.$$

(a) Rewrite this eigenvalue problem in the Sturm-Liouville form;

(b) Prove that its eigenfunctions of different eigenvalues are orthogonal to each other under a certain weight. What is this weight?

(c) Determine these eigenvalues and eigenfunctions.